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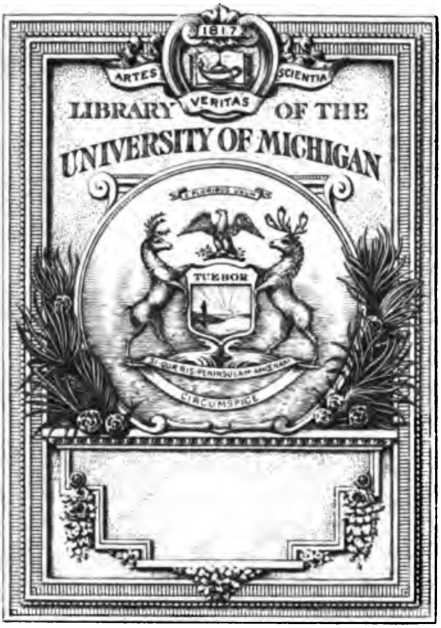
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THE
LAWS of CHANCE:
OR,
A Mathematical INVESTIGATION
OF THE
PROBABILITIES ariſing from any
propoſed Circumſtance of Play.

APPLIED TO

The SOLUTION of a great Variety of
PROBLEMS relating to CARDS, BOWLS,
DICE, LOTTERIES, &c.

By *SAMUEL CLARK*,
Teacher of MATHEMATICS.

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THE

PREFACE.

AS *the* Laws of Chance *have been treated of by ſeveral of the moſt eminent Mathematicians of the preſent age, it may perhaps be deemed either impertinent or needleſs to offer any thing more upon the ſame ſubject. But as my chief deſign was rather to render the following ſheets uſeful, by elucidating what has been done by others, than by any additions of my own, yet notwithſtanding which I flatter myſelf that the Reader will meet with ſeveral things therein contained, not to be found in other authors who have wrote upon this ſubject, I hope I may in ſome meaſure ſtand excuſed for their publication. Of the very few books that have been publiſhed upon this ſubject in the* Engliſh *language, Mr* Abraham Demoivre*'s* Doctrine of Chances *is generally received as the beſt; but as there are ſeveral very curious general ſolutions in that treatiſe, that are rather too abſtruſe to be readily underſtood by thoſe who are not far advanced in the knowledge of the Mathematics, and alſo ſome other eſſential parts, ſuch as the ſummation of ſeries, finding fluents,* &c. *whoſe inveſtigations*

gations are omitted by that learned author. I have in the enſuing pages endeavoured to obviate thoſe difficulties, by conſtantly obſerving to make the ſolutions of ſome eaſy caſes of a problem, preceed the general ſolution, and by giving the inveſtigations of ſuch theorems and lemmas as are made uſe of in order to facilitate the underſtanding of this work. How far I have ſucceeded in this attempt, I leave to the determination of the Reader, who I hope will eaſily be inclined to pardon any inaccuracies as to the ſtyle and manner in which I have wrote, eſpecially when I aſſure him that I have honeſtly done my beſt endeavours to render what is here publiſhed as eaſy to be underſtood, as the nature of the ſubject would admit.

ERRATA [Corrected in Book]

Page 26, line 10, *for* $\sqrt[3]{2}$, *read* $\sqrt[n]{2}$.
P. 35, l. 3, *for* 16, *read* 1,6.
P. 37, l. 13, *for* three games, *read* one game.
P. 55, l. 2 from the bottom, *for* $\frac{n}{n-1}$, *r.* $\frac{n}{2n-1}$.
P. 56, l. 1, *for* $\frac{n}{n-1}$, *read* $\frac{n}{2n-1}$.
P. 58, l. 17, $\frac{n-2}{4n-2}$ muſt be multiplied by $\frac{n-1}{2n-2}$.
P. 101, l. 4, *for* 2, 3, 4, 5, *read* $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$.
P. 103, l. 10, *for* $2 \times \overline{a+b}^x$, *read* $\frac{1}{2} \times \overline{a+b}^x$.
P. 107, l. 7, *for* $+\frac{p^4}{4}$, *read* $-\frac{p^4}{4}$.
P. 149, l. 4, *for* $a+b^p$, *read* $\overline{a+b}^p$.

THE
Laws of Chance.

The INTRODUCTION.

ARTICLE I.

THE probability of an event happening is to the probability of its failing, as the number of different ways by which it can happen, to the number of different ways by which it can fail. Thus if I have 4 Chances for winning 10 pounds, and 4 chances whereby I may not win any thing, my expectation will in this caſe be worth 5 pounds, it being an equal chance whether I get 10 pounds or nothing; and conſequently, if a perſon was to purchaſe my expectation, he ought to give me 5 pounds for it. Again, ſuppoſe a perſon holds a certain ſum of money in each hand, and I am to chuſe which hand I will, I ſay the value of my expectation is in this caſe half the ſum of money held in both hands: for ſuppoſe 5 pounds in one hand and 7 in the other, then it is evident I have an equal chance for either 5 pounds or 7, wherefore my expectation is evidently worth 6 pounds.) If I have 5 chances to gain 10 pounds and 3 chances to fail thereof, then the value of my expectation will ariſe by dividing 10 pounds into two parts in the ratio of 5 to 3, and taking the greateſt part, *viz.* 6*l.* 5*s.* which is the value required. If a, c, e, g, are the chances which I have for the ſums b, d, f, h, reſpectively, then the value of my expectation will be expreſſed by $\frac{ab+cd+ef+gh}{a+c+e+g}$, for $\frac{ab}{n}$ $(n=a+c+e+g)$ is my expectation upon the ſum b, $\frac{cd}{n}$ my expectation upon the ſum d, and ſo on upon the reſt of the ſums, whence my total expectation will be $\frac{ab+cd+ef+gh}{n}$ or reſtoring the value of n, $\frac{ab+cd+ef+gh}{a+c+e+g}$.

ARTICLE II.

BUT if we conſider the ratio of the probability of an event happening, to that of its failing, abſtracted from the value of the ſum to be won or loſt thereby, then ſuch probability may be expreſſed by a fraction whoſe numerator conſiſts of the chances for its happening, and denominator of the chances whereby it may both happen and fail, thus, if an event has 3 chances to happen and 2 to fail, the fraction $\frac{3}{5}$, will repreſent the probability of its happening, and that fraction whoſe numerator conſiſts of the chances whereby it may fail, and denominator of the chances whereby it may both happen and fail, will repreſent the probability of the event failing, that is $\frac{2}{5}$, in the caſe propoſed.

ARTICLE III.

THE fractions which repreſent the probabilities of happening and failing being added together will always make 1 or unity, as the probabilities $\frac{3}{5}+\frac{2}{5}=1$.

ARTICLE IV.

THE expectation upon any propoſed ſum, is eſtimated by the product of that ſum by the probability of obtaining it.

ARTICLE V.

EVENTS are ſaid to be independent when they have no connexion with one another, and that the happening or failing of ſome of them does not affect the happening or failing of the others.

ARTICLE VI.

THOSE events are ſaid to be dependent when the probability of any of them happening or failing, affects the probabilities of the remaining ones.

ARTICLE VII.

THE probability of two events (independent of each other) happening, is equal to the product of the probabilities whereby thoſe events may happen ſingly. Suppoſe with a common die of ſix faces I undertake to throw the ace twice ſucceſſively, and in caſe I ſucceed to receive a certain ſum S, imagine I have already thrown the ace once, then I have one chance for the ſum S and 5 for nothing, whence the value of my expectation after having thrown the ace once is $\frac{1}{6} \times S$ (*Art.* I.) But before I begin to throw I have one chance for $\frac{1}{6} \times S$ and 5 for nothing, whence $\frac{1}{6} \times \frac{1}{6} \times S$ is (*Art.* I.) the value of my expectation upon the ſum S, and conſequently $\frac{1}{6} \times \frac{1}{6}$ is the probability of obtaining the ſum S. By the ſame way of reaſoning, the probability of throwing three aces ſucceſſively with a ſingle die is found to be $\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}$.

Corollary.

Since the probability of happening of two or three events may be conſidered as compounded of the reſpective probabilities of their happening ſeparately, it follows that the probability of the happening of any number of events, *i. e.* that they ſhall all happen, is equal to the product of the probabilities of thoſe events happening conſidered ſingly. If inſtead of throwing an ace three times ſucceſſively with a ſingle die as in the example above, it had been required to throw three aces in one throw with three dice, the probability would have been exactly the ſame: and here let it be obſerved that in many other caſes where one event is or may be conſidered as compounded of ſeveral others more ſimple, which are all decided at the ſame time, as in throwing of dice, *&c.* it will facilitate the reaſoning very much to ſuppoſe them determined one by one in a ſucceſſive order.

ARTICLE VIII.

THE probability of the happening of two events (dependent on each other) is equal to the product of the probability of the happening of one of them, by the probability which the other will have of happening when the firſt ſhall have been conſidered as having happened. Suppoſe that out of one ſingle heap of 13 cards of one colour, it ſhould be required to take out the ace in the firſt place, and then the duce, and that it were required to aſſign the probability of doing it. Imagine that I am in caſe I ſucceed to receive the ſum S, and ſuppoſe that I have already taken out the ace and were to draw the ſecond time, it is evident that there are 12 cards left, out of which there are 11 againſt me and but 1 for me, therefore my expectation is in this caſe $\frac{1}{12} \times S$ (*Art.* IV.) therefore before I begin to draw I have one chance for $\frac{1}{12} \times S$, and 12 chances for nothing, whence my expectation is truly expreſſed by $\frac{1}{13} \times \frac{1}{12} \times S$, and $\frac{1}{13} \times \frac{1}{12}$ is the required probability. From hence it may be inferred that the probability of the happening of two events dependent, is equal to the product of the probability of one of them happening by the probability of the other happening, when the firſt ſhall have been conſidered as having happened, and the ſame rule will extend to the happening of any given number of aſſigned events.

We have ſeen before how to determine the probability of the happening or failing of as many events independent as may be aſſigned, we have likewiſe ſeen in this laſt article how to determine the probability of the happening of as many events dependent as may be aſſigned. But to determine the probability that of given events dependent, ſome of them ſhall happen and at the ſame time, the probability of failing of ſome others, being a matter of ſome difficulty, will be ſhewn farther on.

ARTICLE IX.

IF I have ſeveral expectations upon ſeveral ſums, it is very plain that my expectation upon the whole is the ſum of the expectations I have upon the particulars. Thus ſuppoſe there are two events, ſuch that the firſt has 3 chances to happen and 2 to fail, the ſecond event has 4 chances to happen and 5 to fail, and that I be to receive the ſum of 40 pounds in caſe the firſt ſucceeds, and to a like ſum of 40 pounds in caſe the ſecond ſucceeds alſo. The value of my expectation is required. In the firſt caſe I have 3 chances for 40 pounds and 2 chances for nothing, my expectation in this circumſtance is (*Art.* I.) worth $\frac{3}{5}\times40$, and upon the ſecond event I have 4 chances for 40 pounds and 5 chances for nothing, my expectation in this caſe is worth $\frac{4}{9}\times40$ (*Art.* I.) and conſequently my total expectation is worth $\frac{3}{5}\times40+\frac{4}{9}\times40$ or $41\frac{7}{9}$ pounds. But on the other hand if I am to receive 40 pounds in caſe that one or other of the two aforeſaid events ſhall happen, then the method of ſolution will be ſomewhat different, becauſe under this conſideration my expectation on the ſecond event will ceaſe upon the happening of the firſt, and that therefore this expectation takes place only in caſe the firſt doth chance to fail. Now if the firſt has failed then I have 4 chances for 40 pounds and 5 chances for nothing, whence the value of my expectation is $\frac{4}{9}\times40$, wherefore before either event is determined my expectation will be worth $\frac{8\times40}{5\times9}+\frac{3\times40}{5}$ or $\frac{7}{9}$ of 40 pounds, that is $31\frac{1}{9}$ pounds. The ſame might have been determined otherwiſe, thus, The probability of the firſt event failing is $\frac{2}{5}$ (*Art.*II.) and that of the ſecond happening $\frac{4}{9}$, whence the probability that the firſt ſhall fail and the ſecond happen is $\frac{2}{5}\times\frac{4}{9}$ (*Art.* VII.) But if the firſt ſhould happen, of which the probability is $\frac{3}{5}$, I am intitled to the ſum depending, therefore my total probability is $\frac{2}{5}\times\frac{4}{9}+\frac{3}{5}$ or $\frac{7}{9}$, and my total expectation $\frac{7}{9}\times 40$ pounds, or $31\frac{1}{9}$ (*Art.* IV.) the ſame as before determined.

In order to make the preceeding Articles familiar it may not be amiſs to apply them to the ſolution of ſome eaſy Problems, ſuch as are the following.

PROBLEM I.

To find the probability of throwing an ace in one throw with two common dice.

SOLUTION.

Imagine the ace excluded from each die, then there will be but five faces upon each die, and conſequently all the different ways, that they can come up, *viz.* 25, will be the chances for miſſing an ace, which ſubtracted from 36, all the chances upon two dice, leaves 11, the number of chances for throwing an ace in one throw and the probability $\frac{11}{36}$. If we ſuppoſe the dice to be thrown ſingly the problem may be ſolved thus, If I throw an ace the firſt throw of which the probability is $\frac{1}{6}$ I win, and conſequently have no occaſion to throw again. But if I miſs the ace the firſt throw, of which the probability is $\frac{5}{6}$, I muſt to win throw it in the ſecond; wherefore the pobability of miſſing it the firſt throw and throwing it the ſecond being $\frac{5}{6}\times\frac{1}{6}$ (*Art.* VI.) it follows that $\frac{5}{6}\times\frac{1}{6}+\frac{1}{6}$ or $\frac{11}{36}$ is the total probability, the ſame as before.

Corollary.

Hence it is evident that let the number of dice be what they will, the chances for miſſing an ace each throw with thoſe dice, will be expreſſed by the number of different ways that the ſaid dice, each having one face leſs than before, can come up; that is, if n be the number of dice to be thrown, f the faces upon each, then the probability of miſſing an ace each throw will be $\frac{\overline{f-1}^n}{f^n}$ and that of throwing it $1-\frac{\overline{f-1}^n}{f^n}$ (*Art.* III.) or $\frac{f^n-\overline{f-1}^n}{f^n}$.

PROBLEM II.

To find the probability of throwing one ace and not more, in one throw with 4 dice.

SOLUTION.

Conceive each ace upon any three of the dice expunged, then the throw will be made with 4 dice, three of them having no ace, and the other die one. Now with (125 the cube of 5) all the ways that the three dice ſo reduced can come up, the ace of the other may be combined, but as the ace of any other die might have been retained, it follows that the number of dice multiplied by 125, gives the number of chances for throwing one ace preciſely, *viz.* 500, whence the probability is $\frac{500}{1296}$. Or it may be found thus, The probability of throwing an ace the firſt throw, and miſſing it three times ſucceſſively afterwards with a common die, is $\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}$ (*Corol. Art.* VII.) But becauſe the probability will be exactly the ſame for throwing an ace the ſecond, third, or fourth time and miſſing it the others, it follows that the required probability will be $4\times\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{500}{1296}$ the ſame as before.

Corollary.

If the number of chances for throwing one ace preciſely, *viz.* 500, be ſubtracted from 1296, all the chances upon 4 dice, the remainder 796, will be thoſe for not throwing one ace preciſely, and are alſo thoſe for throwing either no ace, or two at leaſt, whence if it were required to find the probability of throwing two aces at leaſt in one throw with

4 dice, we need only ſubtract the chances for miſſing one ace, *viz.* 625 from 796, and we have 171 for the required chances, $\frac{171}{1296}$ the probability.

PROBLEM III.

To find the probability of throwing two aces preciſely, in one throw with 4 common dice.

SOLUTION.

Let us ſuppoſe the aces on 2 dice taken away, and on the other 2 only the aces left, then in each throw we ſhall be certain to have 2 aces come up. Now the 2 dice without aces may come up 25 ways, with each of which the 2 aces muſt come up alſo. But as we were at liberty to reduce any other 2 dice, it follows that as many different pairs as can be taken in 4, that is 6, juſt ſo often muſt 25, the chances upon the dice without aces, be repeated to give thoſe required, equal to 150, whence the required probability is $\frac{150}{1296}$.

Corollary.

It is evident that 1296—150, leaves the chances for miſſing two aces preciſely, that is 1146, theſe contain thoſe for throwing either no ace, one, or three at leaſt, whence to find the chances for throwing three aces at leaſt, in one throw with 4 common dice, we muſt from 1146 ſubtract the ſum of the chances for miſſing all the aces, and throwing one preciſely, that is 1125, wherefore the chances required are 21, and the probability $\frac{21}{1296}$.

PROBLEM IV.

Let there be a heap of 16 counters, whereof 6 are red and 10 black ones, what is the probability that in drawing two of them blindfold, they ſhall be both red ones.

SOLUTION.

Suppoſe them taken one by one. If the firſt drawn ſhould be a red one, of which the probability is $\frac{6}{16}$, the probability upon the ſecond will be $\frac{5}{15}$, and that of both theſe events happening $\frac{6}{16}\times\frac{5}{15}$ (*Art.* VII.) or $\frac{30}{240}$, the probability ſought.

If the agreement be ſuch, that either the firſt or ſecond counter drawn happening to be red, the drawer ſhould be intitled to a certain ſum S, his expectation may be found thus, If the firſt drawn counter is a red one, his expectation upon the ſecond will entirely vaniſh, becauſe drawing one red counter inſures to him the propoſed ſum, but if a black one ſhould be drawn firſt, of which the probability is $\frac{10}{16}$, his expectation upon the ſecond drawing will be $\frac{6}{15} \times S$, whence his total expectation is $\overline{\frac{6}{16} + \frac{10}{16} \times \frac{6}{15}} \times S$ or $\frac{5}{8} \times S$. The probability of ſucceeding may be determined with more facility by finding the probability of drawing two black counters ſucceſſively, and ſubtracting it from unity, the remainder being certainly the probability of drawing either both red counters or one of each colour, but whichſoever of theſe ſhall happen, the drawer wins. Now the probability of drawing two black counters ſucceſſively is $\frac{10}{16} \times \frac{9}{15}$, which ſubtracted from unity leaves $\frac{15}{24}$, or $\frac{5}{8}$, this multiplied by S, gives the value of the drawer's expectation, the ſame as before.

PROBLEM V.

Let there be a lottery conſiſting of 100 tickets, wherein there are four prizes; to find the probability that in the three firſt tickets that are drawn, there ſhall be one prize at leaſt.

SOLUTION.

Here it is evident that if we find the probability of the three firſt drawn tickets being all blanks, and ſubtract it from unity, the remainder will be the probability of drawing one prize at leaſt. The probability of drawing a blank the firſt time is $\frac{96}{100}$. If a blank ſhould be ſo drawn, there then remains 95 blanks, and the probability of drawing the next ticket a blank is $\frac{95}{99}$, that of the third ticket being a blank $\frac{94}{98}$, whence $\frac{96}{100}\times\frac{95}{99}\times\frac{94}{98}$, is (*Cor. Art.* VII.) the probability of the three firſt drawn tickets being blanks, and therefore $1-\frac{96}{100}\times\frac{95}{99}\times\frac{94}{98}$ or $\frac{941}{8085}$ is the required probability, or that of taking one or more prizes in the three firſt drawn tickets.

Corollary.

Hence it appears, that in many caſes it will be much more convenient to inveſtigate the probability of the contrary to what is required in the problem to happen, rather than the probability of what is ſpecified therein.

PROBLEM VI.

A perſon with a ſingle die undertakes to throw the ace twice, before either the duce or tray once; required the proportion of the odds againſt him?

SOLUTION.

There being two chances againſt him, *viz.* duce and tray, only one for him, the ace, his probability of bringing up the ace at any ſignificant throw will be $\frac{1}{3}$, and that of ſucceeding twice ſucceſſively, will be $\frac{1}{3}\times\frac{1}{3}$; or $\frac{1}{9}$, hence the odds are as 8 to 1.

PROBLEM VII.

Suppoſe an heap of 13 cards of one colour, and another heap of 13 cards of another colour; what is the probability that in taking one card at a venture from each heap, I ſhall take the two aces?

SOLUTION.

The probability of taking the ace from the firſt heap is $\frac{1}{13}$. Now it being very plain that the taking or not taking the ace from the firſt heap has no influence in the taking or not taking the ace from the ſecond, it follows therefore, that ſuppoſing that ace taken out, the probability of taking the ace from the ſecond heap will alſo be $\frac{1}{13}$, theſe two events being independent, the probability of their both happening will (*Art.* VII.) be $\frac{1}{13}\times\frac{1}{13}$, or $\frac{1}{169}$.

ARTICLE X.

Let x denote the probability of happening of one event, y that of a ſecond, z of a third, then it will follow that $1-x$, $1-y$, $1-z$, will repreſent the reſpective probabilities of their failing.

Let it be required to find the probability of the happening of them all, it is plain that by what has been already demonſtrated, the anſwer will be $x \times y \times z$. If it be required to find the probability of their all failing, the anſwer will be $\overline{1-x} \times \overline{1-y} \times \overline{1-z}$. Let it be propoſed to aſſign the probability of ſome one of them, or more, happening. As this queſtion is equivalent to this other, What is the probability of their not all failing, the anſwer will be $1-\overline{1-x} \times \overline{1-y} \times \overline{1-z}$. Let it be demanded what is the probability of the happening of the firſt and ſecond, and at the ſame time of the failing of the third, the anſwer will be $x \times y \times \overline{1-z}$. And for the ſame reaſon the probability of the happening of the ſecond and third, and the failing of the firſt, will be $yz \times \overline{1-x}$. And again, for the probability of the happening of the firſt and third, and the failing of the ſecond, the anſwer will be $xz \times \overline{1-y}$, and the ſum of theſe three probabilities, viz. $x \times y \times \overline{1-z} + yz \times \overline{1-x} + xz \times \overline{1-y}$ will expreſs the probability of the happening of any two of them, but of no more than two.

being drawn one by one, the probability of taking A firſt will be $\frac{1}{n}$, that of taking B next $\frac{1}{n-1}$, of C next $\frac{1}{n-2}$, *&c.* to *(p)* terms, *(p)* being the number of letters or things to be taken, whence (by *Corol. Art.* VII.) $\frac{1}{n}\times\frac{1}{n-1}\times\frac{1}{n-2}\times\frac{1}{n-3}$, *&c.* *(p)* is the probability that the *(p)* letters ſhall be drawn in the order aſſigned, *viz.* A firſt, B ſecond, C third, *&c.*

Corollary II.

If all the letters or things be to be drawn, then $p=n$, and the expreſſion becomes $\frac{1}{n\times\overline{n-1}\times\overline{n-2}\ \&c.\ (n)}$ or $\frac{1}{1\times2\times3\times4\times5\ (n)}$ therefore ſince there is but one way for all the letters, or things repreſented by them, to come out in the required order, it follows, that the denominator of the above fraction contains all the different ways that the ſaid *(n)* letters can be varied, and is commonly called the number of changes or permutations of the given quantities.

PROBLEM II.

Any number *(n)* of things, as A, B, C, D, E, *&c.* being given, to find the probability that in taking any number *(p)* of them as it happens, they ſhall be the *(p)* firſt of the foregoing order, without any regard to order between the quantities drawn.

SOLUTION.

Let the things be drawn one by one, then becauſe in the whole there are *(p)* aſſigned ones, the probability of drawing one of thoſe will be $\frac{p}{n}$, if one ſhould be ſo drawn, there remains $n-1$ things in all, of which $p-1$ are of the aſſigned ones, the probability of drawing one of thoſe next will therefore be $\frac{p-1}{n-1}$, and that of both theſe events happening $\frac{p}{n}\times\frac{p-1}{n-1}$, ſuppoſe this has happened, there then remains $n-2$ things in all, of which $p-2$ are of the aſſigned ones, and conſequently the probability of taking one of thoſe next will be $\frac{p-2}{n-2}$, whence (by *Corollary Art.* VII.) $\frac{p}{n}\times\frac{p-1}{n-1}\times\frac{p-2}{n-2}$, will be the probability of ſucceeding three times ſucceſſively, and from hence the law of continuation is manifeſt, *viz.* $\frac{p}{n}\times\frac{p-1}{n-1}\times\frac{p-2}{n-2}\times\frac{p-3}{n-3}$, *&c.* *(p)* wherein it is viſible, that let p be what it will, the numerator of the laſt factor will be 1, therefore the expreſſion becomes — $\frac{1\times2\times3\times4\ldots(p)}{n\times\overline{n-1}\times\overline{n-2}\,(p)}$, the probability required, which taken from 1 leaves $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\ \&\ldots(p)-1\times2\times3\ldots(p)}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\ \&\ \ldots\ldots\ldots(p)}$ the probability of not taking (p) aſſigned things, conſequently the proportion between the number of ways by which the event may happen and fail, is as $1\times2\times3\times4$, &c. (p) to $n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}$, &c. $(p)-1\times2\times3\times4$, &c. (p) Now as there is only one way by which the event can happen, thoſe for its failing will be $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}(p)}{1\times2\times3\times4\times5\times6\ (p)}-1$, therefore $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}(p)}{1\times2\times3\times4\times5\times6\ (p)}$ will denote the number of different ways by which (n) things may be taken p, by p, and is commonly called the combinations of quantities.

Otherwiſe.

Suppoſe as before the things to be drawn ſingly, and that I had already taken $p-1$ of the aſſigned ones, it is evident $n-p+1$, are thoſe remaining of the whole number (n), and therein I have one chance to win, or to obtain a certain ſum S, and $n-p$ chances for nothing, the value of my expectation is in this caſe (by *Art.* I.) $\frac{S}{n-p+1}$. If I have proceeded right for only $p-2$ times of drawing, then the value of my expectation is (by

Art. I.) $\frac{1\times 2\times S}{\overline{n-p-1}\times\overline{n-p-2}}$. By the ſame way of reaſoning, before I begin to draw, my expectation will be worth $\frac{\overline{1\times 2\times 3\ldots p}\times S}{\overline{n-p-1}\times\overline{n-p-2}\ldots(p)}$ whence the number of combinations of (n) things taken p, by p, is equal to $\frac{\overline{n-p+1}\times\overline{n-p+2}\times\overline{n-p+3},\ \text{\&c.}\ldots(p)}{1\times 2\times 3\times 4\times 5\times 6\times 7,\ \text{\&c.}\ldots(p)}$.

Example.

How many different pairs or combinations of two, can be made with four aces. Here uſing the firſt theorem, we have $\frac{n}{1}\times\frac{n-1}{2}=6$, n being 4 and $p=2$, or by the ſecond theorem $\frac{\overline{n-p+1}\times\overline{n-p+2}}{1\times 2}=6$, the ſame as before.

PROBLEM III.

Let there be a given number of things, whereof ſome are ſeveral times repeated, for inſtance, *a*, *a*, *b*, *b*, *c*, *c*, *c*, *c*, *c*; what is the probability that by drawing theſe letters, or things repreſented by them, one by one, until they are all exhauſted, they ſhall come out in the very ſame order as they are here placed, that is, that all the *a*'s ſhall come out firſt, the *b*'s next.

SOLUTION.

There being nine letters in all, and two of them *a*'s, the probability of the *a*'s coming out before any one of the other letters is $\frac{2}{9}\times\frac{1}{8}$, that of drawing the *b*'s next $\frac{2}{7}\times\frac{1}{6}$, wherefore the probability of drawing the *a*'s firſt, and the *b*'s next, is $\frac{2}{9}\times\frac{1}{8}\times\frac{2}{7}\times\frac{1}{6}$, or $\frac{1}{756}$. If this happens, there is no need of proceeding any farther, for the taking *c*'s next is a certainty.

Corollary.

If there be ſeveral ſorts of things mixed together, as *p* of the firſt ſort, *q* of the ſecond, *r* of the third, *s* of the fourth, *&c.* and the number of things ſo mixed be called *n*, the probability of the ſaid ſorts coming out in the propoſed order, will be equal to —

$$\frac{1\times2\times3,\ \&c.\ (p)\times1\times2\times3\times4,\ \&c.\ (q)\times1\times2\times3\times4\times5,\ \&c.}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\overline{n-6}\times\overline{n-7}\times}$$

$$\frac{(r)\times1\times2\times3\times4,\ \&c.\ (s)}{\overline{n-8}\times\overline{n-9},\ \&c.}$$

wherein the numerator conſiſts of as many parts as there are ſorts to be taken, and the denominator of as many factors, as is denoted by the number of factors in the whole numerator.

Example.

Let there be 20 counters of the ſame ſhape and ſize, whereof three are marked each *a*, four marked each *b*, and four each marked *c*, the remaining nine each marked *d*. It is required to find the probability that drawing them one by one, all the *a*'s ſhall come out firſt, the *b*'s next, and the *c*'s third. Here $n=20$, $p=3$, $q=4$, $r=4$, whence the expreſſion for the required probability is $\frac{1\times2\times3\times1\times2\times3\times4\times1\times2\times3\times4}{20\times19\times18\times17\times16\times15\times14\times13\times12\times11\times10}$ equal to $\frac{1}{1939938000}$.

PROBLEM IV.

If A and B play together with ſingle bowls, and ſuch be the ſkill of A, that he knows by experience he can give B two games out of three; what is the proportion of their ſkill, or what are the odds that A may get any one game aſſigned.

SOLUTION.

Since A can upon equality of chance give B two games out of three, it is evident A muſt undertake to win three games before B wins one. Let A's chances for winning any aſſigned game be *(a)* thoſe for the contrary *(b)*, then his probability of winning the firſt game will be $\frac{a}{a+b}$ (*Art.* I.) and that of winning three games ſucceſſively $\frac{a}{a+b}\times\frac{a}{a+b}\times\frac{a}{a+b}$ (*Cor. Art.* VII.) But as they are ſuppoſed to play upon equal terms, the laſt determined probability muſt be equal to $\frac{1}{2}$, (*Art.* II.) whence the equation $\frac{a^3}{\overline{a+b}^3}=\frac{1}{2}$. Conſequently $b : a :: \sqrt[3]{2}-1 : \sqrt[3]{1}$, or as 50 to 13 nearly.

Corollary.

If A undertakes to win (n) times before B wins once, upon equality of chance, then A may juſtly lay the odds of 1 to $\sqrt[n]{2}-1$, that he wins any game aſſigned.

PROBLEM V.

A has two chances to beat B, and B has one chance to beat A, but there is one chance that intitles them both to withdraw their own ſtake, ſuppoſe S; it is required to find the gain of A.

SOLUTION.

It is evident that A has two chances to gain 2 S, one chance to obtain S, and one chance not to receive any thing, whence (by *Art.* I.) $\frac{5S}{4}$ is the value of his expectation, from which ſubtracting his ſtake S, the remainder $\frac{S}{4}$, is his gain. But if the ſaving clauſe, or that chance that intitles A to withdraw his ſtake be excluded, then he has 2 chances for 2 S, and 1 chance for nothing, conſequently his expectation is worth $\frac{4S}{3}$, from which ſubtract his ſtake S, there remains $\frac{S}{3}$, his gain. But to make the ſolution more general, let A and B be ſuppoſed to depoſit the ſum *(s)*, and let *(a)* repreſent the chances A has to beat B, and *(b)* thoſe which B has to beat A. Suppoſe alſo there are *(m)* chances common to both, that is, for A to take ſuch a part of the whole ſum laid down, as is expreſſed by $\frac{p}{r}$, B to have the remainder. Now A has *(a)* chances for 2 *s* *(b)* chances for nothing, and *(m)* chances for $\frac{p}{r}\times 2s$, whence (by *Art.* I.) the value of his expectation is $\frac{2asr+2mps}{ar+br+mr}$, which leſſened by *(s)* his ſtake leaves his gain $\frac{sra+2mps-brs-mrs}{\overline{a+b+m}\times r}$. If we ſuppoſe no common chance, then A's gain will be $\frac{a-b}{a+b}\times s$. If it were required to aſſign the ratio of p to r, ſo that the gain of A ſhall be the ſame in both caſes, then $\frac{ra+2mp-br-mr}{ar+br+mr}$ muſt be equal to $\frac{a-b}{a+b}$, from which equation we get, by proper reduction, $\frac{p}{r}=\frac{a}{a+b}$, hence we may infer that if the parts of the whole ſum depoſited, which A and B are reſpectively to take up, upon the happening of the ſaving chances *(m)*, are in the proportion of *a* to *b*, then the common chance gives no advantage to A, above what he would have had, if thoſe chances had not exiſted.

PROBLEM VI.

There are two parcels of three cards, the firſt containing king, queen and knave of hearts; the ſecond parcel the king, queen and knave of diamonds. Now ſuppoſe I am promiſed the ſum S in caſe that in taking a card out of each parcel, I ſhould take out either the king of hearts, or the king of diamonds, it is required to find the value of my expectation.

SOLUTION.

Imagine I have drawn a card from one heap, and miſſed taking out a king, then I have 1 chance to draw a king from the other heap, whereby I get the ſum S, and 2 chances to miſs drawing the king, whereby I do not get any thing, in this caſe my expectation is worth $\frac{1}{3}$S, ſo that before I begin to draw I have 1 chance for S, and 2 for $\frac{1}{3}$S, whence $\frac{5S}{9}$ is the value of my expectation.

The ſame otherwiſe.

The probability of taking the king of hearts from the firſt parcel is $\frac{1}{3}$, if this ſhould not happen, of which the probability is $\frac{2}{3}$, I have ſtill the chance for ſucceeding in the ſecond parcel, the probability of which is $\frac{1}{3}$, wherefore $\frac{2}{3}\times\frac{1}{3}$ is the ſecond part of my probability, which added to $\frac{1}{3}$, my firſt part, gives $\frac{5}{9}$ my total probability, hence $\frac{5}{9}\times$S is the value of my expectation (by *Art.* IV.)

PROBLEM VII.

Suppoſe the number of chances for an event happening to be *(a)*, thoſe for its failing *(b)*, in any one trial; it is required to find the probability of the event happening preciſely *(p)* times, in a given number *(n)* of trials.

SOLUTION.

Becauſe the event is to happen *(p)* times, and not more, in *(n)* trials, it is evident whenever it has happened *(p)* times, the remaining trials muſt fail, otherwiſe the event does not happen *(p)* times preciſely in *(n)* trials. The probability of the event happening in any one trial is $\frac{a}{a+b}$, and that of its happening (p) times ſucceſſively $\frac{a^p}{\overline{a+b}^{\,p}}$. The probability of its failing $(n-p)$ trials, is $\frac{b^{n-p}}{\overline{a+b}^{\,n-p}}$, therefore $\frac{a^p}{\overline{a+b}^{\,p}}\times\frac{b^{n-p}}{\overline{a+b}^{\,n-p}}$, is the probability that the event ſhall happen all the firſt (p) trials, and fail all the remaining trials. But there being the very ſame probability for its happening any other (p) aſſigned trials, and failing the reſt, it follows, that as many different ways as p things can be taken or combined in (n) things, juſt ſo often muſt the ſaid probability be repeated, to give the required one. The combinations of (p) things in (n) is $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, &c. (p) (by *Prob.* II.) whence $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, &c. (p) $\times\frac{a^p\times b^{n-p}}{\overline{a+b}^{\,n}}$, is the probability ſought.

Example I.

A perſon with five common dice, undertakes to throw one ace and no more, at the firſt throw; what are the odds againſt him? By ſuppoſing the dice thrown ſeparately, and the five throws as ſo many trials, we have $n=5$, $p=1$,

$a=1$, $b=5$, and $\frac{n}{1}\times\frac{a^{p}\times b^{n-p}}{\overline{a+b}^{n}}=\frac{3125}{7776}$, the odds as 4651 to 3125.

Example II.

What is the probability that in throwing up five halfpence there ſhall come up two heads preciſely? Here $a=1$, $b=1$, $p=2$, $n=5$, and $\frac{n}{1}\times\frac{n-1}{2}\times\frac{a^{p}\times b^{n-p}}{\overline{a+b}^{n}}=\frac{5}{16}$, the probability required.

PROBLEM VIII.

If the number of chances for an event happening, and failing, be *(a)* and *(b)* reſpectively; what is the probability that the event happens three times at leaſt, in five trials.

SOLUTION.

The probability that the event ſhall happen once preciſely is (by the laſt *Problem*) $\frac{5b^{4}a}{\overline{a+b}^{5}}$, therefore $1-\frac{5b^{4}a}{\overline{a+b}^{5}}$, is the probability of the contrary. The probability that the event ſhall happen twice is (by the laſt *Problem*) $\frac{10a^{2}b^{3}}{\overline{a+b}^{5}}$, and the probability that the event ſhall not happen at all is $\frac{b^{5}}{\overline{a+b}^{5}}$. Now it is very evident that the ſum of the two laſt probabilities being ſubtracted from 1 — $\frac{5b^4a}{\overline{a+b}}$, leaves $1 - \frac{5b^4a}{\overline{a+b}^5} - \frac{10a^2b^3}{\overline{a+b}^5} - \frac{b^5}{\overline{a+b}^5}$, or its equal $\frac{a^5+5a^4b+10a^3b^2}{\overline{a+b}^5}$ for the probability that the event happens three times or oftener, in 5 trials; wherein we may obſerve that the numerator conſiſts of the three firſt terms of the binomial $a+b$, raiſed to the fifth power, and alſo that the exponents of the ſingle powers of *(a)* are no where to be found leſs than 3, the number that denotes the times (at leaſt) the event is to happen; conſequently the remaining terms of the binomial, will expound the number of chances for the event not happening ſo often as 3 times in 5 trials, and univerſally if *(n)* be the aſſigned number of trials, *(p)* the number of times the event is to happen at leaſt, in thoſe trials, the probability of ſucceeding will be found, by raiſing $a+b$ to the *(n)* power, from thence ſelecting all the terms wherein the exponent of *(a)* is either equal to, or greater than *(p)*, and dividing their ſum by $\overline{a+b}^n$. The remaining terms of the binomial ſo raiſed, or thoſe where the exponents of the ſingle powers of *(a)* are leſs than *(p)* being added together, and their ſum divided by $\overline{a+b}^n$, will give the probability that the event ſhall not happen ſo often as *(p)* times in *(n)* trials.

Example I.

A with four common dice undertakes to throw two aces at leaſt, in one throw; required his probability of doing it. Here if the number of dice be conſidered as ſo many trials, we have $n=4$, $p=2$, wherefore $a+b$ being raiſed to the fourth power, and from thence the three firſt terms being ſelected, we have $\frac{a^4+4a^3b+6b^2a^2}{\overline{a+b}^4}$ for the probability, equal to $\frac{171}{1296}$, a being 1, and b, 5.

Example II.

Suppoſing one to throw up nine halfpence, what are the odds that there ſhall come up more than three heads. Here $n=9$, $a=1$, $b=1$, $p=4$, wherefore taking thoſe terms of $a+b$, raiſed to the ninth power, where the exponent of the ſingle powers of *(a)* is either equal to, or greater than 4, will give $1+9+36+84$ $(=130)$ which being divided by 512 $(\overline{1+1}^9)$ quotes $\frac{65}{256}$ for the probability of bringing up four heads or more, whence the required odds as 191 to 65.

PROBLEM IX.

If A can without advantage, or diſadvantage, give B one game out of three; what are the odds that A ſhall take any one game aſſigned, or in other terms, what is the proportion of their ſkill.

SOLUTION.

Becauſe A can without any diſadvantage give B one game out of three, he muſt, to beat B, win three games before B wins two, from whence it appears, that the play will end in four games at moſt, for if A gets two and B one, the next game will evidently determine the play in favour of one of them, wherefore ſeeing the play will neceſſarily be ended in four games, let *(a)* and *(b)* be their reſpective chances for winning any aſſigned one. The probability of A's winning three times at leaſt in four trials is $\frac{a^4+4a^3b}{\overline{a+b}|^4}$ (by the laſt *Prob.)* and that of B's winning twice or oftner, in four trials, is $\frac{b^4+4b^3a+6b^2a^2}{\overline{a+b}|^4}$ theſe probabilities, becauſe they play upon equality of chance, muſt be equal to each other, hence this equation $a^4+4a^3b=b^4+4b^3a+6b^2a^2$, let $a:b::1:z$, then

az

$a z = b$, ſubſtitute $a z$ for b in the equation, it becomes $1 + 4 z = z^4 + 4 z^3 + 6 z^2$ ſolved $z = 1{,}6$ nearly, whence a to b as 5 to 8.

PROBLEM X.

A and B, whoſe proportion of ſkill, or chances for winning any aſſigned game, are as (a) to (b) play together, and they find that A wants 5 games of being up, and B 3; what are the reſpective probabilities of winning?

SOLUTION.

It is evident the play will be ended in 7 games at moſt, for ſuppoſe A gets 4, and B 2, then they want only 1 game each of being up, therefore the next will terminate the play: this being premiſed, if A wins 5 games in 7, he beats B, of which the probability is $\frac{a^7 + 7 a^6 b + 21 a^5 b^2}{\overline{a+b}^7}$, and $1 - \frac{a^7 + 7 a^6 b + 21 a^5 b^2}{\overline{a+b}^7}$ or $\frac{35 a^4 b^3 + 35 a^3 b^4 + 21 a^2 b^5 + 7 b^6 a + b^7}{\overline{a+b}^7}$ is B's probability of beating A, or of getting 3 games before A gets 5.

PROBLEM XI.

A and B, whoſe chances for winning any aſſigned game are as (a) to (b), having played together ſome time, find that the former wants (p) games of being up, and the latter (q); what are their reſpective probabilities of winning.

SOLUTION.

The play will be ended in $p+q-1$ games at moſt, for ſuppoſe A has got $p-1$ games, and B $q-1$ games, then they each want one of being up, and conſequently the next game will terminate the play: hence $p-1+q-1+1$ or $p+q-1$, are the games at the end of which, either A or B will be up. Put $p+q-1=n$, then if A wins (p) or a greater number of games in (n) B loſes, of this happening the probability will be found equal to

$$\frac{a^n+na^{n-1}b+n\times\frac{n-1}{2}\times a^{n-2}b^2+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times a^{n-3}b^3 \,..\, a^p\times b^{n-p}}{\overline{a+b}^{\,n}},$$

which ſubtracted from unity, leaves ——

$$\frac{b^n+nb^{n-1}a+n\times\frac{n-1}{2}\times b^{n-2}a^2 \ldots\ldots b^q\times a^{n-q}}{\overline{a+b}^{\,n}},$$ the probability that B wins the ſet.

PROBLEM XII.

Three gameſters, A, B, and C, play together on this condition, that he ſhall win the ſet who has ſooneſt got a certain number of games, the proportion of the chances which each of them has to get any one game aſſigned, or which is the ſame thing, the proportion of their ſkill being reſpectively as a, b, c: now after they have played ſome time, they find themſelves in this circumſtance, that A wants 1 game of being up, B 2 games of being up, and C 3, the whole ſtake between them being 1; what is the expectation of each.

SOLUTION.

It is certain that A's total probability will be expreſſed by the ſum of all the different probabilities which he has for winning ~~three~~ one game~~s~~ before either B wins two, or C three. B's total probability, by the ſum of all the different probabilities which he has for winning two games before either A wins one, or C three, and C's total probability, by the ſum of all the different probabilities which he has for winning three games before either A wins one, or B two. Wherefore the probabilities of thoſe events happening, that muſt neceſſarily take place, in order that the players may ſucceed ſingly, being found and added together, will give three ſuch ſums as will expreſs their required probabilities, thoſe ſums being each multiplied by the whole ſtake, gives their reſpective expectations. But in order to render this as clear as poſſibly I could, I have annexed the three following tables, wherein are placed the neceſſary events, and their reſpective probabilities of happening, that muſt determine the expectations of the gameſters.

TABLE I. *for* A.

Ways of winning.	*Probabilities.*
A the firſt time,	$\frac{a}{d}$
B firſt, A ſecond,	$\frac{b}{d}\times\frac{a}{d}$
C firſt, A ſecond,	$\frac{c}{d}\times\frac{a}{d}$
C 1ſt and 2d, A 3d,	$\frac{c}{d}\times\frac{c}{d}\times\frac{a}{d}$
C 1ſt, B 2d, A 3d,	$\frac{c}{d}\times\frac{b}{d}\times\frac{a}{d}$
B 1ſt, C 2d, A 3d,	$\frac{b}{d}\times\frac{c}{d}\times\frac{a}{d}$
B 1ſt, C 2d & 3d, A 4th	$\frac{b}{d}\times\frac{c}{d}\times\frac{c}{d}\times\frac{a}{d}$
C 1ſt & 2d, B 3d, A 4th	$\frac{c}{d}\times\frac{c}{d}\times\frac{b}{d}\times\frac{a}{d}$
C 1ſt, B 2d, C 3d, A 4th	$\frac{c}{d}\times\frac{b}{d}\times\frac{c}{d}\times\frac{a}{d}$

TABLE II. *for* B.

Ways of winning.	*Probabilities.*
B 1st and 2d time,	$\frac{b}{d}\times\frac{b}{d}$
C 1st, B 2d and 3d,	$\frac{c}{d}\times\frac{b}{d}\times\frac{b}{d}$
C 1st & 2d, B 3d & 4th,	$\frac{c}{d}\times\frac{c}{d}\times\frac{b}{d}\times\frac{b}{d}$
C 1st, B 2d, C 3d, B 4th,	$\frac{c}{d}\times\frac{b}{d}\times\frac{c}{d}\times\frac{b}{d}$
B 1st, C 2d, B 3d,	$\frac{b}{d}\times\frac{c}{d}\times\frac{b}{d}$
B 1st, C 2d & 3d, B 4th,	$\frac{b}{d}\times\frac{c}{d}\times\frac{c}{d}\times\frac{b}{d}$

TABLE III. *for* C.

Ways of winning.	*Probabilities.*
C 1st, 2d & 3d times,	$\frac{c}{d}\times\frac{c}{d}\times\frac{c}{d}$
C 1st, 2d & 4th, B 3d,	$\frac{c}{d}\times\frac{c}{d}\times\frac{c}{d}\times\frac{b}{d}$
C 1st, 3d & 4th, B 2d,	$\frac{c}{d}\times\frac{c}{d}\times\frac{c}{d}\times\frac{b}{d}$
C 2d, 3d & 4th, B 1st,	$\frac{c}{d}\times\frac{c}{d}\times\frac{c}{d}\times\frac{b}{d}$

In the firſt column of the firſt table, are all the different ways by which A can poſſibly win, in the ſecond column of the ſame table are inſerted the probabilities of thoſe events happening, (d being equal to $a+b+c$) and the ſame is to be underſtood of the ſecond and third tables, which are for B and C. Now by taking the ſum of the probabilities in the firſt table, we have the probability that A wins the ſet, equal to $\frac{a}{d}+\frac{ab+ac}{dd}+\frac{2abc+acc}{d^3}+\frac{3abc^2}{d^4}$, by ſumming up the probabilities in the ſecond table we get $\frac{bb}{dd}+\frac{2bbc}{d^3}+\frac{3bbc^2}{d^4}$, for B's probability of winning the ſet; and laſtly, by taking the ſum of all the probabilities in the third table, we have $\frac{c^3}{d^3}+\frac{3bc^3}{d^4}$ for C's. Now in order to ſave the trouble of conſtructing ſuch tables, when a problem of this ſort is to be ſolved, we muſt endeavour to obtain the law by which the probabilities of the gameſters may be continued, when their number, and that of the games they reſpectively want of being up, are any given numbers at pleaſure. For the law of continuation in the problem before us, I write $\frac{a}{d}$ for the firſt term of A's probability, then I multiply $b+c$ by $a+b+c$, in the product $ab+ac+bb+2bc+cc$, the quantities $ab+ac$, and bb being divided each by dd, make the ſecond term in A's probability, and the firſt in B's. The remaining part $2bc+cc$ of the product I multiply by $a+b+c$, from the product $2abc+acc+2bbc+3bcc+c^3$, ſelect $2abc+ac^2$, $2cb^2$, and c^3, each of theſe quantities being divided by d^3, give the third term in A's probability, the ſecond in B's, and firſt in C's reſpectively. Laſtly, I multiply the remaining part of the product by $a+b+c$, then if each new term of this product be divided by d^4, there will ariſe $\frac{3abcc}{d^4}$, $\frac{3bbcc}{d^4}$, $\frac{3bc^3}{d^4}$ for the fourth term in A's probability, the third in B's, and ſecond in C's. Having proceeded thus far, we may eaſily obſerve, that in A's probability the ſingle powers of (a) only are to be found, and in B's probability the ſecond power of (b) is found. Laſtly, in C's only the third power of (c) is to be found, reſpectively anſwering to the games they want of being up. The law of continuation in the denominators is alſo viſible, being d, d^2, d^3, d^4, &c. until the exponent of d becomes equal to the number of games beyond which the play cannot be protracted.

The number of gameſters and games wanting, being other numbers than thoſe given in this problem, if tables be in like manner conſtructed for their probabilities of winning the ſet, we may, by comparing thoſe probabilities with what has been already delivered, obtain a general method of ſolution, as follows.

Let the gameſters be A, B, C, D, &c. the games they want of being up $p, q, r, s,$ &c. their chances for winning any aſſigned game, in the proportion of $a, b, c, d,$ &c. and put $a+b+c+d,$ &c. $=b$, this being done, raiſe $a+b+c+d,$ &c. to that power whoſe index is equal to the leaſt of the given numbers $p, q, r, s,$ &c. or to one of the leaſt, if the two leaſt are equal to one another, as (p), and from that power take out the term wherein the exponent of the correſpondent quantity (a) is equal to (p), and if there are any terms wherein the indices of (b) and (c), &c. are equal to $q, r, s,$ &c. take thoſe terms alſo, and having divided each of the terms ſo taken by b^p, place the ſeveral quotients each in different columns marked A, B, C, D, &c. ſo that the ſaid power of (a) may be in the column A, that of (b) (if any) in the column B, &c. Now having proceeded ſo far, multiply the remaining terms of the ſaid power by $a+b+c$, *&c.* and from the product ſelect all the terms where the indices of the powers of a, b, c, *&c.* are reſpectively equal to p, q, r, *&c.* and having divided each by b^{p+1}, place them according to the foregoing method, in the columns A, B, C, *&c.* Let the laſt remainder be multiplied in like manner ſtill by $a+b+c$, *&c.* and ſelect again from the product the terms wherein the exponents of a, b, c, *&c.* are reſpectively equal to p, q, r, *&c.* and having divided each by b^{p+2}, let the quantities be diſpoſed of as before, and proceed in this manner, repeating the operation till all the terms are exhauſted, then the quantities that are at laſt found in the columns A, B, C, D, *&c.* will be the probabilities of winning reſpectively.

Example.

Let there be four gameſters, A,B,C,D, let the proportion of their ſkill be as 1, 1, 2, 3, and ſuppoſe A wants 1 game of being up, B 2, C 3, D 2. The general theorem, *viz.*

A.

$$\text{A.}\left\{\begin{array}{l}\frac{a}{h}+\frac{ad+ab+ac}{h^2}+\frac{acc+2abd+2adc+2abc}{h^3}+\\ \frac{3adcc+3abcc+6abcd}{h^4}+\frac{12abdcc}{h^5}.\end{array}\right.$$

$$\text{B.}\quad \frac{bb}{h^2}+\frac{2bbc+2dbb}{h^3}+\frac{6bbcd+3bbcc}{h^4}+\frac{12bbdcc}{h^5}.$$

$$\text{C.}\quad \frac{c^3}{h^3}+\frac{3bc^3+3dc^3}{h^4}+\frac{12bdc^3}{h^5}.$$

$$\text{D.}\quad \frac{dd}{h^2}+\frac{2bdd+2cdd}{h^3}+\frac{3ddcc+6bcdd}{h^4}+\frac{12bddcc}{h^5}.$$

Will give $\frac{6465}{16807}$, $\frac{1313}{16807}$, $\frac{1352}{16807}$, $\frac{7677}{16807}$, for the probabilities of A, B, C, D, reſpectively, for in this caſe $a=1$, $b=1$, $c=2$, $d=3$, and $h=7$.

PROBLEM XIII.

If A and B equally ſkillful, play with two bowls each; it is required to find the probability that one of them (ſuppoſe A) ſhall get both bowls at an end, and alſo the probability of his getting preciſely one bowl at an end.

SOLUTION.

If A gets both bowls at an end, his firſt bowl muſt be nearer the jack than B's, of which the probability is $\frac{1}{2}$. Otherwiſe it is impoſſible he can win both bowls at the end. Now A muſt with his other bowl, not only beat B's ſecond bowl, but alſo that which he has already thrown, the probability of doing it is $\frac{1}{3}$, therefore $\frac{1}{2}\times\frac{1}{3}$ or $\frac{1}{6}$ is (*Art.* VII.) the probability of A's getting both bowls at an end. Let x be the probability that A ſhall get preciſely one bowl at an end, then as $\frac{5}{6}$, which is the probability that he doth not win both bowls, certainly contains thoſe for his winning either one bowl preciſely, or none, it follows, that $\frac{5}{6}-\frac{1}{6}-x=x$, whence $x=\frac{1}{3}$, the probability that A wins preciſely one bowl at an end.

PROBLEM XIV.

If A and B with equal ſkill play together, each with three bowls; it is required to determine the probability that either of them has for winning all the three bowls of the other.

SOLUTION.

The probability that A ſhall have one bowl nearer the jack, than any one of B's is $\frac{1}{2}$; ſuppoſe this has happened, then A may be ſuppoſed to have two bowls, and B three. Now if one of A's bowls beats B's three, of which the probability is $\frac{2}{5}$, then A will have two bowls nearer the jack than any one of B's, whence A's probability of winning two bowls or more at the end, is $\frac{1}{2} \times \frac{2}{5}$, if this ſhould happen, then as he has one bowl left, and the probability of its beating B's three, being $\frac{1}{4}$, it follows, that (*Art.* VII.) $\frac{1}{2} \times \frac{2}{5} \times \frac{1}{4}$, is the probability that A has for winning the three bowls at the end, and B's probability for winning A's bowls is the ſame.

PROBLEM XV.

A and B upon equality of ſkill play together, each having a certain number (*n*) of bowls; required the probability that one of them, as A, ſhall win any aſſigned number of bowls or more, at an end. And alſo, the probability that he wins any aſſigned number preciſely, at an end.

SOLUTION.

Suppoſe one of A's bowls nearer the jack than B's, of this happening the probability is $\frac{1}{2}$. The bowls that A has left is $n-1$, and it is evident that one or more of thoſe muſt beat all B's, (for A to have two or more neareſt the jack) the probability of this is $\frac{n-1}{2n-1}$, therefore $\frac{1}{2} \times \frac{n-1}{2n-1}$, is the probability that A ſhall have two bowls or more, neareſt the jack. Imagine this has happened, now the probability that ſome other of his bowls ſhall be nearer the jack than any of B's is $\frac{n-2}{2n-2}$, whence the probability that A ſhall get three bowls or more at the end, is $\frac{1}{2}\times\frac{n-1}{2n-1}\times\frac{n-2}{2n-2}$. And by the very ſame way of reaſoning, his probability of winning four bowls or more, at an end, will be found equal to $\frac{1}{2}\times\frac{n-1}{2n-1}\times\frac{n-2}{2n-2}\times\frac{n-3}{2n-3}$, whence the law of the progreſs is manifeſt. The probability that A ſhall have one bowl neareſt the jack is $\frac{1}{2}$, that of his having two is $\frac{n-1}{2n-1}$, as found above, whence the probability that he ſhall have one bowl and not more, neareſt the jack, is evidently $\frac{1}{2}-\frac{n-1}{2n-1}$ or $\frac{1}{2}\times\frac{n}{2n-1}$. In the ſame manner, by ſubtracting his probability for having three bowls neareſt the jack, from that of his having two ſo, the remainder, $\frac{1}{2}\times\frac{n}{2n-1}\times\frac{n-1}{2n-2}$, will be the probability that A ſhall have two bowls and not more, nearer the jack than any of B's. Whence the probability of getting any aſſigned number *(p)* of bowls preciſely, at an end, is $\frac{1}{2}\times\frac{n}{2n-1}\times\frac{n-1}{2n-2}\times\frac{n-2}{2n-3}\times\frac{n-3}{2n-4}$, *&c.* continued to $(p+1)$ factors.

PROBLEM XVI.

If A with *(m)* bowls, and B with *(n)*, play together upon equality of ſkill; it is required to find their reſpective probabilities for winning any aſſigned number *(r)* of bowls, or more, at an end, as alſo of winning that number preciſely, at an end.

SOLUTION.

The probability that A ſhall have one bowl or more neareſt the jack, is $\frac{m}{m+n}$ (*Art.* I.) or $\frac{m}{s}$ (putting $s=m+n$) if this ſhould happen, then A may be ſuppoſed to have $m-1$ bowls left, and all the bowls left between them $s-1$, whence the probability that A ſhall have a ſecond bowl nearer the jack, than any one of B's, will be $\frac{m-1}{s-1}$, and that of his having two at leaſt neareſt the jack, will be $\frac{m}{s}\times\frac{m-1}{s-1}$, in the like manner the probability that A ſhall win three bowls or more, at an end, is $\frac{m}{s}\times\frac{m-1}{s-1}\times\frac{m-2}{s-2}$, from whence the law of continuation is viſible, it being $\frac{m}{s}\times\frac{m-1}{s-1}\times\frac{m-2}{s-2}\times\frac{m-3}{s-3}$, &c. *(r)*. By writing *(n)* for *(m)* in the ſeries, we ſhall have $\frac{n}{s}\times\frac{n-1}{s-1}\times\frac{n-2}{s-2}\times\frac{n-3}{s-3}$, &c. *(r)* for B's probability of winning *(r)* bowls, or more, at an end. The probability that A gets one bowl or more, at the end is $\frac{m}{s}$, that he gets two or more at the end is $\frac{m}{s} \times \frac{m-1}{s-1}$, the difference between $\frac{m}{s}$ and $\frac{m}{s} \times \frac{m-1}{s}$, that is $\frac{m}{s} \times \frac{n}{s-1}$ is the probability of A's winning one bowl preciſely at the end, and that of his winning two bowls preciſely at the end, is $\frac{m}{s} \times \frac{m-1}{s-1} \times \frac{n}{s-2}$, and univerſally we have $\frac{m}{s} \times \frac{m-1}{s-1} \times \frac{m-2}{s-2} \times \frac{m-3}{s-3}$, *&c.* $(r) \times \frac{n}{s-r}$ for the probability that A ſhall get (r) bowls preciſely at the end, $\frac{n}{s} \times \frac{n-1}{s-1} \times \frac{n-2}{s-2} \times \frac{n-3}{s-3}$, *&c.* $(r) \times \frac{m}{s-r}$ that of B's getting (r) bowls preciſely at the end.

PROBLEM XVII.

A and B, whoſe proportion of ſkill is as *a* to *b*, play together, each having (n) bowls; to find the probability that A has of getting any number (p) of bowls, or more, at an end, alſo that of his getting preciſely that number at an end.

SOLUTION.

Since the proportion of ſkill ſignifies the odds that any aſſigned bowl of A's ſhall be nearer the jack than any aſſigned bowl of B's, we may ſuppoſe, that each of A's bowls may contain (a) chances, and each of B's (b) chances, then (na) expounds the whole number of chances in all A's bowls, and (nb) the number of chances in all B's, this being premiſed, A's probability of having one bowl at leaſt nearer the jack than any one of B's, will be $\frac{na}{na+nb}$ or (putting $d=\frac{na+nb}{a}$) $\frac{n}{d}$, if this ſhould happen, then A having $n-1$ bowls left, and the chances therein being $\overline{n-1}\times a$, the probability of his having a ſecond bowl nearer the jack, than any one of B's will be $\frac{a+\overline{n-1}}{an+bn-a}$, or $\frac{n-1}{d-1}$, that of his having two, or more, neareſt the jack will be $\frac{n}{d}\times\frac{n-1}{d-1}$, (*Art.* VII.) and by the ſame ſort of proceſs we get $\frac{n}{d}\times\frac{n-1}{d-1}\times\frac{n-2}{d-2}$, for the probability that A may have three, or a greater number of bowls, nearer the jack than any one of B's. From hence it will appear, that $\frac{n}{d}\times\frac{n-1}{d-1}\times\frac{n-2}{d-2}\times\frac{n-3}{d-3}$, &c. (p) will denote A's probability of having (p) or a greater number of bowls, nearer the jack than any one of B's, and the probability of A having (p) bowls preciſely nearer the jack than any one of B's will be $\frac{n}{d}\times\frac{n-1}{d-1}\times\frac{n-2}{d-2}\times\frac{n-3}{d-3}$, &c. (p) —

$\frac{n}{d} \times \frac{n-1}{d-1} \times \frac{n-2}{d-2} \times \frac{n-3}{d-3}$, &c. $(p+1)$ or its equal, $\frac{nb}{da} \times \frac{n}{d-1} \times \frac{n-1}{d-2} \times \frac{n-2}{d-3} \times \frac{n-3}{d-4}$, &c. $(p+1)$. Becauſe, if from A's probability of having (p) or more bowls neareſt the jack, there be taken that of his having $(p+1)$ or more bowls neareſt the jack, the remainder muſt certainly be the probability of getting (p) bowls preciſely: or the ſame thing may be determined otherwiſe, as thus, when (p) of A's bowls are nearer the jack than any one of B's, if then B gets the next bowl, A will have (p) bowls and not more, neareſt; the former of theſe probabilities is $\frac{n}{d} \times \frac{n-1}{d-1} \times \frac{n-2}{d-2} \times \frac{n-3}{d-3}$, &c. (p); and the latter $\frac{nb}{\overline{d-p} \times a}$, their product $\frac{n}{d} \times \frac{n-1}{d-1} \times \frac{n-2}{d-2} \times \frac{n-3}{d-3}$, &c. $(p) \times \frac{nb}{\overline{d-p} \times a}$, or its equal, $\frac{nb}{da} \times \frac{n}{d-1} \times \frac{n-1}{d-2} \times \frac{n-2}{d-3}$, &c. $(p+1)$ is the probability required, the ſame as before.

PROBLEM XVIII.

A and B each having two bowls, play together upon equality of ſkill, what are their reſpective probabilities of winning, when A wants 1 bowl of being up, B 2?

SOLUTION.

If A gets one bowl or more the firſt end, he wins the game; but if B gets one bowl preciſely the firſt end, they will each want one of being up, the probability that A obtains this laſt mentioned circumſtance of play is $\frac{1}{3}$, which multiplied by $\frac{1}{2}$ gives $\frac{1}{6}$, for one part of his probability for being up before B, to which add $\frac{1}{2}$, that of his getting one bowl or more the firſt end, the ſum $\frac{2}{3}$ is his probability of getting one bowl before B gets 2. Hence B's probability of beating A is $\frac{1}{3}$.

PROBLEM XIX.

The other things remaining as in the laſt problem. Let A want 1 bowl of being up, B 3; to determine their reſpective probabilities of winning.

SOLUTION.

Firſt, ſuppoſe A wins the firſt end, in this caſe he beats B, the probability of this happening is $\frac{1}{2}$. Secondly, if B gets juſt one bowl at the firſt end, then B will want two bowls of being up, in this caſe A's probability will be $\frac{2}{3}$, wherefore the probability that A beats B, when this has happened, is $\frac{1}{3} \times \frac{2}{3}$ or $\frac{2}{9}$. But if B ſhould get two bowls and not more the firſt end, they will each want one of being up, and A's probability is then $\frac{1}{2}$; but as B's probability of getting into this circumſtance of play is $\frac{1}{6}$, it follows, that $\frac{1}{2} \times \frac{1}{6}$ will be A's probability in this laſt caſe, and that $\frac{1}{2} + \frac{2}{9} + \frac{1}{2} \times \frac{1}{6}$ is that of his being up before B, whence B's probability required is $1 - \frac{1}{2} - \frac{2}{9} - \frac{1}{2} \times \frac{1}{6}$, or $\frac{7}{36}$.

PROBLEM XX.

If A and B play together with a certain number *(n)* of bowls each; what are their probabilities of winning, when A wants one bowl of being up, and B two?

SOLUTION.

If B wins juſt one bowl at the firſt end, then they are upon equality of chance; the probability of this happening is $\frac{1}{2} \times \frac{n}{2n-1}$ (*Prob.* XV.) If he wins two or more, the firſt end, of which the probability is $\frac{1}{2} \times \frac{n-1}{2n-1}$, he wins the game, therefore the former probability being multiplied by $\frac{1}{2}$, and added to the latter, gives $\frac{1}{4} \times \frac{n}{2n-1} + \frac{1}{2} \times \frac{n-1}{2n-1}$ or $\frac{3n-2}{8n-4}$, B's probability for winning the game, which taken from unity leaves A's $= \frac{5n-2}{8n-4}$.

Corollary.

If $n=1$, or they play with a ſingle bowl each, their probabilities will be $\frac{3}{4}$ and $\frac{1}{4}$ reſpectively. If with two bowls each, $\frac{8}{12}$, $\frac{4}{12}$ reſpectively. If $n=3$, the ratio between their chances, will be that of 13 to 7. Here we may obſerve, that as the number of bowls increaſe, the ratio between their chances decreaſes, but that ratio will never deſcend ſo low as 5 to 3, becauſe each multiple of n is leſſened by the number 2.

PROBLEM XXI.

If A and B, each having (n) bowls, play together upon equality of ſkill, and A wants one bowl of being up, B wants (p) bowls of being up; it is required to find their reſpective probabilities of winning.

SOLUTION.

Firſt, ſuppoſe B wants three bowls of being up. If B gets one bowl preciſely, the firſt end, his probability of winning will be $\frac{3n-2}{8n-4}$ (*Prob.* XX.) which multiplied by $\frac{1}{2}\times\frac{n}{2n-1}$, that of his getting one bowl preciſely, the firſt end, gives $\frac{1}{2}\times\frac{n}{2n-1}\times\frac{3n-2}{8n-4}$, the firſt part of B's required probability. But if he gets two bowls preciſely, the firſt end, they will in that caſe be upon equal terms, whence $\frac{1}{2}$, which is B's probability in that circumſtance, multiplied by $\frac{1}{2}\times\frac{n}{2n-1}\times\frac{n-1}{2n-2}$, that of its taking place, the product, $\frac{1}{4}\times\frac{n}{2n-1}\times\frac{n-1}{2n-2}$ is the ſecond part of B's required probability; to the ſum of theſe two parts add the probability of his getting three bowls or more, the firſt end, the ſum, *viz.* $\frac{n^2-n}{\overline{8n-4}\times\overline{2n-2}}+\frac{3n^2-2n}{\overline{8n-4}\times\overline{4n-2}}+\frac{\overline{n-1}\times\overline{n-2}}{\overline{4n-2}\times\overline{2n-2}}$, is B's probability for winning, and being deducted from unity, leaves A's.

Secondly, let B want four bowls of being up. If he gets one bowl preciſely, at the firſt end, he will want three of being up. Let his probability for winning, in this circumſtance, as determined above, be called P, then if P be multiplied by $\frac{1}{2}\times\frac{n}{2n-1}$, the probability of obtaining it, the product $P\times\frac{1}{2}\times\frac{n}{2n-1}$ is the firſt part of B's probability of winning. If he gets two bowls, and not more, at the firſt end, of which the probability is $\frac{1}{2}\times\frac{n}{2n-1}\times\frac{n-1}{2n-2}$, we have (by putting Q for his probability of winning, if that happens) $\frac{1}{2} \times \frac{n}{2n-1} \times \frac{n-1}{2n-2} \times Q$ for the ſecond part of B's probability of winning. If he gets three bowls preciſely, the firſt end, his probability of winning will be $\frac{1}{2}$, which multiplied by $\frac{1}{2} \times \frac{n}{2n-1} \times \frac{n-1}{2n-2} \times \frac{n-2}{2n-3}$, that of its taking place, gives the third part of his required probability, to the ſum of theſe parts add the probability of his getting four bowls or more the firſt end, the total will give B's probability for being up before A, equal to $\frac{1}{2} \times \frac{n}{2n-1} \times P + \frac{1}{2} \times \frac{n}{2n-1} \times \frac{n-1}{2n-2} \times Q + \frac{1}{4} \times \frac{n}{2n-1} \times \frac{n-1}{2n-2} \times \frac{n-2}{2n-3} + \frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2} \times \frac{n-3}{2n-3}$. Hence by writing P, Q, R, S, T, *&c.* for B's reſpective probabilities for winning, when he wants 1, 2, 3, 4, 5, *&c.* bowls of being up, and multiplying thoſe quantities by their reſpective probabilities of taking place, we ſhall obtain the following products, *viz.* $\frac{1}{2} \times \frac{n}{2n-1} \times \frac{n-1}{2n-2} (p+1) \times P + \frac{1}{2} \times \frac{n}{2n-1} \times \frac{n-1}{2n-2} (p-1) \times Q + \frac{1}{2} \times \frac{n}{2n-1} \times \frac{n-1}{2n-2} (p-2) \times R + \frac{1}{2} \times \frac{n}{2n-1} \times \frac{n-1}{2n-2} (p-3) \times S$, *&c.* to theſe add $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2} \times \frac{n-3}{2n-3} (p)$, the ſum, is the probability that B has for getting (p) or a greater number of bowls, before A gets one.

Corollary.

If $n = 1$, or they play with one bowl each, B's probability will be expreſſed by $\frac{3n^2-2n}{\overline{8n-4}\times\overline{4n-2}} = \frac{1}{8}$, the other terms vaniſhing. It may be here obſerved, that B's probability in this caſe might be contracted into $\frac{9n^2-13n+4}{8\times\overline{2n-1}^2}$, but this expreſſion, when $n = 1$, becomes equal to nothing, inſtead of $\frac{1}{8}$; whereas being left unreduced, is equally true in all caſes.

PROBLEM XXII.

A and B, each having *(n)* bowls, play together upon equality of ſkill, A wants three bowls of being up, and B two; it is required to determine their reſpective probabilities of winning.

SOLUTION.

If A gets one bowl preciſely, the firſt end, they will each want two bowls of being up, if this happens, A's probability will be $\frac{1}{2}$, which multiplied by that of its happening, *viz.* $\frac{1}{2}\times\frac{n}{2n-1}$, gives $\frac{1}{4}\times\frac{n}{2n-1}$, the firſt part of his required probability. If A gets exactly two bowls, the firſt end, he will want one of being up, the probability of his getting into this circumſtance is $\frac{1}{2}\times\frac{n}{2n-1}\times\frac{n-1}{2n-2}$, that of winning after this has happened $\frac{5n-2}{8n-4}$ (by *Prob.* XX.) whence the ſecond part of A's probability is $\frac{1}{2}\times\frac{n}{2n-1}\times\frac{n-1}{2n-2}\times\frac{5n-2}{8n-4}$. Now if B gets one bowl, and not more, the firſt end, then $\frac{1}{2}\times\frac{n}{2n-1}\times\frac{n^2-n}{\overline{8n-4}\times\overline{2n-1}}+\frac{1}{2}\times\frac{n}{2n-1}\times\frac{3n^2-2n}{\overline{4n-2}\times\overline{8n-4}}$ $+\frac{\overline{n-1}\times\overline{n-2}}{\overline{4n-2}\times\overline{2n-2}}\times\frac{1}{2}\times\frac{n}{2n-1}$, will be the third part of A's probability. Wherefore, if to the ſum of theſe probabilities already found, we add $\frac{1}{2}\times\frac{n-1}{2n-1}\times\frac{n-2}{2n-2}$, which is A's probability of getting three bowls or more, at the firſt end, we have A's total probability required, which taken from unity leaves B's. If we put $P=\frac{5n-2}{8n-4}$, $Q=\frac{n^2-n}{\overline{8n-4}\times\overline{2n-2}}+\frac{3n^2-2n}{\overline{4n-2}\times\overline{8n-4}}+\frac{\overline{n-1}\times\overline{n-2}}{\overline{4n-2}\times\overline{2n-2}}$, the above expreſſion for A's probability, becomes $\frac{n}{8n-4}+\frac{n\times\overline{n-1}}{\overline{4n-4}\times\overline{2n-1}}\times P+\frac{n}{4n-2}\times Q+\frac{n-2}{4n-2}\times\frac{n-1}{2n-2}$.

Corollary.

If $n=1$, or they play only with ſingle bowls, A's probability for winning the ſet will be $\frac{n}{8n-4}+\frac{n}{4n-2}\times\frac{3n^2-2n}{\overline{4n-2}\times\overline{8n-4}}=\frac{5}{16}$. If $n=2$,

$n = 2$, A's probability will be expreſſed by

$$\frac{n}{8n-4} + \frac{n \times \overline{n-1}}{\overline{4n-4} \times \overline{2n-1}} \times P + \frac{n}{4n-2} \times \frac{n^2-n}{\overline{8n-4} \times \overline{2n-2}} + \frac{3n^2-2n}{\overline{4n-2} \times \overline{8n-4}}, = \frac{37}{108}.$$

PROBLEM XXIII.

Other things remaining as in the laſt problem. Let A want four bowls of being up, and the ſum played for be denoted by 1.

SOLUTION.

Let P be put for A's probability if he ſhould happen to want only one bowl of being up, Q if two, and R if three. Put alſo his probability of getting one, two, and three bowls, preciſely, at an end, equal to *a*, *b*, *&c.* reſpectively. Now if A gets juſt one bowl the firſt end, of which the probability is *(a)*, his expectation upon that event will be aR. Again, if he gets preciſely two bowls the firſt end, his expectation will be bQ, and if he gets preciſely three cP, hence $aR + bQ + cP + d$ (*d* being put for the probability of getting four bowls or more the firſt end) will expreſs that part of his total expectation which depends upon B's not having as yet won any bowls at all. But if B ſhould get one bowl preciſely, the firſt end, of which the probability is (a), A's total probability will be $a\text{R}+b\text{Q}+c\text{P}+d+a\text{S}$ (S being A's expectation in this laſt caſe). By the very ſame way of reaſoning may their expectations, in any other given circumſtance of play, or bowls wanting between A and B, be readily enough found, the values of P, Q, R, S, &c. and thoſe of a, b, c, d, &c. being firſt obtained, by the methods laid down in the preceeding problems, and thence a general method of ſolution may be inveſtigated, by making the ſolution of each inferior caſe, ſubſervient to that which is next more compounded.

PROBLEM XXIV.

A, B, and C, each having a certain number (m) of bowls, play together, upon equality of ſkill; what are their reſpective probabilities of winning, ſuppoſing that A wants one bowl of being up, B one, and C two, the ſum of their ſtakes being 1?

SOLUTION.

If A gets one bowl or more, the firſt end, he will be intitled to the whole ſtake, whence his expectation upon this event is $\frac{1}{3}$; but if C gets one bowl preciſely, the firſt end, of which the probability is $\frac{1}{3}\times\frac{2m}{3m-1}$, they will all three be upon equal terms, and each intitled to $\frac{1}{3}$ of the ſtake, conſequently A's total expectation is $\frac{1}{3}\times\frac{2m}{3m-1}\times\frac{1}{3}+\frac{1}{3}$, which is alſo B's, therefore C's is $1-\frac{2}{3}-\frac{1}{3}\times\frac{2m}{3m-1}\times\frac{2}{3}$. Other things remaining as above, let C want three bowls of being up, then if he gets one bowl preciſely, the firſt end, the players will be in the circumſtance propoſed in the problem, in that caſe A's expectation will be $\frac{1}{3}+\frac{1}{9}\times\frac{2m}{3m-1}$, this multiplied by $\frac{1}{3}\times\frac{2m}{3m-1}$, the probability of obtaining it, gives $\frac{1}{9}\times\frac{2m}{3m-1}+\frac{1}{3}\times\frac{2m}{3m-1}\times\frac{1}{3}\times\frac{2m}{3m-1}\times\frac{1}{3}$, to which add $\frac{1}{3}$, the ſum will be A's required expectation, from whence C's is eaſily obtained. By the ſame way of reaſoning, the ſolution of any other problem of this ſort, as when the number of players and the bowls they want of being up, are any given ones whatſoever, may be eaſily inveſtigated.

PROBLEM XXV.

An even number of perſons C, D, E, F, *&c.* each having two bowls, play together upon equality of ſkill, C is partner with D, E with F, *&c.* it is required to find the probability which each ſide has of winning, in any circumſtance of their play, or bowls wanting of being up.

SOLUTION.

Firſt, ſuppoſe there are only two partners on each ſide, and that one ſide wants one bowl of being up, the other ſide two. Now if two other players A and B, equally ſkilful with the former, each having four bowls, be ſuppoſed to play the game out in their room, the ſolution of this problem will be the very ſame as that of the propoſed. Therefore $\frac{5n-2}{8n-4}$, which is A's probability of getting up before B (by *Prob.* XX.) is to $\frac{3n-2}{8n-4}$, which is B's probability for getting up before A, as 9 to 5, (becauſe $n=4$ in this caſe) and hence the probability that the partners C, D, beat the other two E, F, is denoted by $\frac{9}{14}$, that of the contrary by $\frac{5}{14}$. If there are three partners on each ſide, then for (n) write 6; and the ratio of their probabilities becomes that of 7 to 4.

Let there be ſtill three partners on each ſide, but ſuppoſe one ſide wants three bowls of being up, and the other ſide one. By *Problem* XXI. the probabilities that A and B have for beating each other are $\frac{9n^2-13n+4}{8\times\overline{2n-1}^2}$ and $1-\frac{9n^2-13n+4}{8\times\overline{2n-1}^2}$, and becauſe in this caſe $n=6$, the proportion of the chances that each ſide has of winning is that of 359 to 125. By proceeding in this manner the expectations of the gameſters may be determined in any other circumſtance of the play. But as it would be very troubleſome to calculate the required probabilities by theſe theorems, when each ſide conſiſts of ſeveral partners, and the bowls wanting of being up, on each ſide, are more than has here been exemplified; the following method of approximating, by which the ſolution may be obtained ſufficiently near the truth, not only when the players are two, but a greater number, may be very uſeful; which method I have taken the liberty to tranſcribe from Mr *Simpſon*'s treatiſe upon this ſubject. The firſt of the following Tables ſhews the odds when there are only two players, and the laſt the limits of the leaſt odds, when the players are indefinite, in both caſes the players are ſuppoſed to play with two bowls each.

		Truly.			*Nearly.*		
2	1	2	:	1	2	:	1
3		29	:	7	$4\frac{1}{7}$	:	1
4		95	:	13	$7\frac{1}{5}$	:	1
5		601	:	47	$12\frac{4}{5}$	:	1
3	2	71	:	37	$1\frac{11}{12}$	:	1
4		247	:	77	$3\frac{1}{5}$	:	1
5		272	:	52	$5\frac{1}{5}$	:	1
4	3	607	:	365	$1\frac{2}{3}$	:	1
5		8507	:	3157	$2\frac{1}{3}$	:	1
5	4	13507	:	21485	$1\frac{3}{5}$	:	1

		Truly.			*Nearly.*		
2	1	5	:	3	$1\frac{2}{3}$	:	1
3		23	:	9	$2\frac{5}{9}$	:	1
4		101	:	27	$3\frac{3}{4}$	:	1
5		431	:	81	$5\frac{1}{3}$	:	1
3	2	77	:	51	$1\frac{1}{2}$	:	1
4		175	:	81	$2\frac{1}{6}$	:	1
5		1535	:	513	30	:	1
4	3	601	:	423	$1\frac{3}{7}$	:	1
5		5311	:	2781	$1\frac{9}{10}$	:	1
5	4	18909	:	13859	$1\frac{3}{8}$	:	1

The conſtruction of the firſt table is thus, When A wants one bowl, and B two of being up, it appears by *Prob.* XX. that $5n-2$ to $3n-2$ will be the proportion of their chances, that is 8 to 4, or 2 to 1, when $n=2$.

But if n be very great, the numerical quantity may be rejected as inconſiderable, whence the leaſt odds becomes $5n$ to $3n$, or 5 to 3, which are the reſpective numbers placed in the ſecond table, when one ſide wants two bowls of being up, the other one. When A wants one bowl of being up, and B three, then (by *Cor. Prob.* XXI.) $23n^2-19n+4$ to $9n^2-13n+4$, is the ratio of their chances, that is, as 29 to 7, for the numbers in the firſt table. If (n) be very great, the quantities $4-19n$ and $4-13n$ may be rejected, whence the leaſt odds is $23n^2$ to $9n^2$ or 23 to 9, for the numbers in the ſecond table. In the ſame manner the remaining proportions in the tables were determined. Now by help of theſe tables the odds may be nearly had in this manner, from each of them take the probabilities of winning in the propoſed caſe for which they are calculated, and let thoſe probabilities be denoted by P, and p, reſpectively, and let n be the number of players on each ſide, then will $\frac{n-1}{n}\times P+\frac{p}{n}$ be the probability of winning.

Example I.

Suppoſe the number of players to be four, *viz.* two of a ſide, and that one ſide wants one bowl of being up, the other two; then by the tables the odds will be as 2 to 1, and $1\frac{2}{3}$ to 1, whence $P=\frac{3}{8}$, $p=\frac{1}{2}$, and $\frac{n-1}{n}\times P+\frac{n}{p}=\frac{7}{16}$.

Example II.

Suppoſe three players of a ſide, and that one ſide wants two bowls of being up, the other four; then by the tables the odds will be $3\frac{1}{5}$ to 1, and $2\frac{1}{6}$ to 1, here $P=\frac{6}{19}$, $p=\frac{5}{21}$, whence $\frac{n-1}{n}\times P+\frac{p}{n}=\frac{347}{1197}$, and the odds are as 850 to 347.

Example III.

Let there be four partners on each ſide, and ſuppoſe one ſide wants one bowl of being up, the other four; then by the tables, $P=\frac{9}{32}$, $p=\frac{7}{36}$, and $\frac{n-1}{n}\times P+\frac{p}{n}=\frac{299}{1152}$, and the odds are as 853 to 299.

PROBLEM XXVI.

If A and B, whoſe proportion of ſkill is as *a* to *b*, play together with two bowls each, what are their reſpective probabilities of winning (the whole ſtake being 1) when A wants 1 bowl of being up, and B 2?

SOLUTION.

A may happen to win 1 bowl or more the firſt end, or B may win preciſely 1 bowl at that end, or elſe both. In the firſt caſe B loſes his expectation, in the ſecond they will each want 1 bowl of being up, but in the third caſe B will be intitled to the whole ſtake. Now by *Problem* XVII, A's probability of winning one bowl or more the firſt end, is $\frac{a}{a+b}$, that of B's winning one preciſely at that end, is (by the ſame *Problem*) $\frac{2a}{2a+b} \times \frac{b}{a+b}$, wherefore in this circumſtance of the play, A's expectation is $\frac{a}{a+b}$, and his total probability equal to $\frac{2a^3+5a^2b+b^2a}{2a^3+5a^2b+4b^2a+b^3}$, which taken from unity leaves $\frac{3b^2a+b^3}{2a^3+5a^2b+4b^2a+b^3}$ for B's.

PROBLEM XXVII.

Other things remaining as in the laſt problem, let B want 3 bowls of being up; to find their reſpective probabilities of winning.

SOLUTION.

B may happen to win either one bowl preciſely at the firſt end, or both. If the firſt caſe happens, of which the probability is $\frac{b}{a+b} \times \frac{2a}{2a+b}$, the players will be in the above mentioned circumſtance, that is, A will want one bowl of being up, and B two. Let B's expectation, as there determined, be called P. Now as he is intitled to $\frac{b}{a+b}$ of the whole ſtake, if the ſecond caſe happens, it follows that his total expectation is equal to $\frac{b}{a+b} \times \frac{2a}{2a+b} \times P + \frac{b^2}{a+b \times 2a+b} + \frac{a}{a+b}$.

Several other uſeful theorems relating to bowls, may in like manner be obtained, by obſerving how each of them is deduced from the preceding one.

If we ſuppoſe the players to have each an unequal number of bowls, as alſo an inequality in their ſkill, or that either, or both thoſe inequalities ſhould vary, according to ſome certain given law, or if four or ſix, or any greater even number of gameſters were to play together in partnerſhip, under any circumſtance of bowls wanting on each ſide, and of any given proportion of ſkill, their probabilities might in the ſame manner be determined.

PRO-

PROBLEM XXVIII.

A perſon undertakes in three caſts with a common die, to throw both an ace and a duce, without regard to which ſhall happen to be firſt thrown; required his probability of winning.

SOLUTION.

The probability that he miſſes both the ace and duce, the firſt throw, but brings them up in the two next, is $\frac{2}{3} \times \frac{1}{18}$. The probability that he brings up one of them the firſt throw, and the other in the two next, is $2 \times \frac{1}{6} \times \frac{11}{36}$; therefore his probability of ſucceeding is $\frac{2}{3} \times \frac{1}{18} + 2 \times \frac{1}{6} \times \frac{11}{36}$, equal to $\frac{5}{36}$.

PROBLEM XXIX.

A with two common dice undertakes to throw both the numbers 5 and 7, at three throws; what are the odds againſt him?

SOLUTION.

Imagine a die having 36 equal and ſimilar faces; now as there are 4 chances for 5, and 6 chances for 7, upon two dice, ſuppoſe each of 4 faces of the aſſumed die marked 5, and 6 other faces each marked 7, then the probability of ſucceeding with this die will be exactly the ſame as with two common dice. The probability that A miſſes all the marked faces the firſt throw, and brings up one of each in the two following, is $\frac{48}{36^2}\times\frac{13}{18}$, that of his throwing a marked face of one ſort the firſt throw, and a marked face of the contrary ſort the two next throws, is $\frac{4}{36}\times\frac{396}{1296}+\frac{3}{18}\times\frac{272}{1296}$ to which add $\frac{48}{1296}\times\frac{13}{18}$, the ſum $\frac{4464}{46656}$ or $\frac{31}{324}$ is A's probability of winning, whence the odds againſt him as 293 to 31.

Otherwiſe.

Conceived each face marked 7 upon the aſſumed die excluded, then 30^3 will expreſs all the chances for 7 being reſtrained from appearing, in each of three throws with the aſſumed die unreduced, and therefore $36^3—30^3$, are the chances for throwing one or more of thoſe faces marked 7. By the ſame way of reaſoning, if we imagine the faces marked 5 excluded, we ſhall have $32^3—26^3$, the chances for one or more of thoſe faces marked 7, coming up without any of thoſe marked 5, which being ſubtracted from $36^3—30^3$, the chances for throwing one or more faces marked 7, without any regard to thoſe marked 5, the remainder $36^3—30^3—32^3+26^3$, or 4464, divided by 36^3, is A's probability of winning, *viz.* $\frac{4464}{46656}$, or $\frac{31}{324}$, the very ſame as before.

PROBLEM XXX.

Let there be a ſolid die, having *(n)* ſimilar and equal faces, whereof *(a)* are marked A, *(b)* of them marked B, and *(c)* of them C; what is the probability that in throwing up a given number *(r)* of ſuch dice, ſome of each of the marked faces A, B, C, ſhall be uppermoſt.

SOLUTION.

The number of chances for the faces marked A being reſtrained from appearing, is $\overline{n-a}^r$, therefore $n^r - \overline{n-a}^r$ expreſſes the number of variations whereby one or more faces marked A may be upwards, therefore if all the faces marked B be now reſtrained from appearing, then there being only $n-b$, faces that can come up, we ſhall have $\overline{n-b}^r - \overline{n-a-b}^r$, for the number of variations that can poſſibly be made ſo as to have A up, and B at the ſame time reſtrained from appearing, which being ſubtracted from $n^r - \overline{n-a}^r$, the different ways that A can poſſibly ariſe, when no reſtriction is made upon B, leaves $n^r - \overline{n-a}^r - \overline{n-b}^r + \overline{n-a-b}^r$, the chances for having both A and B upwards, this is evident, for the chances to have A up without reſtriction upon B, certainly contain thoſe for A coming up both with and without B. If now the faces marked C be ſuppoſed reſtrained from appearing, the number of faces upon each die may be conceived reduced to $n-c$, then the laſt expreſſion, by ſubſtituting $n-c$ for n, becomes $\overline{n-c}^r - \overline{n-a-c}^r + \overline{n-a-b-c}^r - \overline{n-b-c}^r$, which expounds the different ways that A and B can ariſe, C being reſtrained from appearing; this taken from $n^r - \overline{n-a}^r$, which are the chances for having A and B up, without reſtriction upon C, leaves

$$n^r - \left\{\begin{array}{c}\overline{n-a}^r\\ \overline{n-b}^r\\ \overline{n-c}^r\end{array}\right\} + \left\{\begin{array}{c}\overline{n-a-b}^r\\ \overline{n-a-c}^r\\ \overline{n-b-c}^r\end{array}\right\} - \overline{n-a-b-c}^r,$$

for the chances that ſome of each of the faces marked A, B, C, may be upwards, and conſequently, the ſaid expreſſion divided by n^r, gives the probability required.

Corollary I.

By the ſame proceſs it will appear that if a, b, c, and d, expreſs the number of aſſigned faces marked A, B, C, and D, reſpectively, the chances for having thoſe faces all uppermoſt, are equal to

$$n^r - \left\{\begin{matrix}\overline{n-a}^r\\ \overline{n-b}^r\\ \overline{n-c}^r\\ \overline{n-d}^r\end{matrix}\right\} + \left\{\begin{matrix}\overline{n-a-b}^r\\ \overline{n-a-c}^r\\ \overline{n-a-d}^r\\ \overline{n-b-c}^r\\ \overline{n-b-d}^r\\ \overline{n-c-d}^r\end{matrix}\right\} - \left\{\begin{matrix}\overline{n-a-b-c}^r\\ \overline{n-a-b-d}^r\\ \overline{n-a-c-d}^r\\ \overline{n-b-c-d}^r\end{matrix}\right\} + \overline{n-a-b-c-d}^r$$

from whence the law of continuation is manifeſt.

Corollary II.

If a, b, c, and d, are equal to each other, the expreſſion for the chances, for having A, B, C, and D, uppermoſt, becomes $n^r - 4 \times \overline{n-a}^r + 6 \times \overline{n-2a}^r - 4 \times \overline{n-3a}^r + \overline{n-4a}^r$, from whence we may obſerve, that the numerical quantities 1, 4, 6, 4, 1, are the unciæ of a binomial raiſed to that power, whoſe index is equal to the number of ſorts A, B, C, D, and, in other caſes, where the number of ſorts A, B, C, D, E, F, *&c.* is denoted by any given quantity *(b)*, the general theorem for the required probability that ſome of the marked faces ſhall be upwards, is —

$$\frac{n^r - b \times \overline{n-a}^r + b \times \frac{b-1}{2} \times \overline{n-2a}^r - b \times \frac{b-1}{2} \times \frac{b-2}{3} \times \overline{n-3a}^r +}{n^r}$$

$$\frac{b \times \frac{b-1}{2} \times \frac{b-2}{3} \times \frac{b-3}{4} \times \overline{n-4a}^r}{n^r}.$$

Corollary

Corollary III.

When (*n*) is a very large number, and (*a*) a very ſmall one, n^{r}, $\overline{n-a}^{r}$, $\overline{n-2a}^{r}$, &c. are nearly in geometrical progreſſion, or equal to n^{r}, $\overline{n-a}^{r}$, $\frac{\overline{n-a}^{2r}}{n^{r}}$, $\frac{\overline{n-a}^{3r}}{n^{2r}}$, $\frac{\overline{n-a}^{4r}}{n^{3r}}$, &c. reſpectively, therefore the general theorem for the probability ſought, becomes $1 - b\times\overline{1-\frac{a}{n}}^{r} + b\times\frac{b-1}{2}\times\overline{1-\frac{a}{n}}^{2r} - b\times\frac{b-1}{2}\times\frac{b-2}{3}\times\overline{1-\frac{a}{n}}^{3r} + b\times\frac{b-1}{2}\times\frac{b-2}{3}\times\frac{b-3}{4}\times\overline{1-\frac{a}{n}}^{4r}$, &c. or $\overline{1-\overline{1-\frac{a}{n}}^{r}}^{b}$, which ſuppoſe equal to P, then

$$r = \frac{Log.\ \overline{1-P^{\frac{1}{b}}}}{Log.\ \overline{1-\frac{a}{n}}}.$$

Example I.

A perſon undertakes in eight throws with a ſingle die, to fling both ace and duce, required his probability of winning. Here $n=6$, $a=1$, $b=1$, $b=2$, $r=8$, whence $\frac{6^{8}-2\times5^{8}+4^{8}}{6^{8}} = \frac{962882}{1679616} = \frac{4}{7}$ nearly, the probability ſought.

Example II.

To find in how many caſts with a ſingle die a perſon may, upon equality of chance, undertake to throw all the ſix faces. Here is given $P = \frac{1}{2}$, $a = 1$, $n = 6$, whence $r = \frac{\text{Log. } \overline{1 - \frac{1}{2}}|^{\frac{1}{6}}}{\text{Log. } 1 - \frac{1}{6}} = 13$ nearly, wherefore in 13 caſts he has the advantage, but in 12 the diſadvantage.

Example III.

To find in how many trials one may with equal chance undertake to throw all the doublets with a pair of common dice. There being 36 chances upon two dice, if we ſuppoſe a ſingle die having 36 faces, and ſix of thoſe marked A, B, C, D, E, F, (becauſe there are ſix doublets upon a pair of common dice) it will be the very ſame thing to throw with this ſingle die, as with the pair, wherefore $n = 36$, $a = 1$, $b = 6$, whence $r = \frac{\text{Log. } \overline{1 - \frac{1}{2}}|^{\frac{1}{6}}}{\text{Log. } 1 - \frac{1}{36}}$ $= 79$.

PROBLEM XXXI.

A and B play together with a pair of common dice, A wins if he throws 7, B if he throws 6; it is required to find the proportion of their chances, ſuppoſing B to have the advantage of the firſt throw, and the play to be continued till one of them wins. The value of the whole ſtake between them being denoted by 1.

SOLUTION.

Let the value of A's expectation be x, then B's is $1-x$, now whenever it is his turn to throw, A's hazard is x, but when it is A's turn to throw it is ſomething leſs, or his expectation is greater, let it be y, now as there are 5 chances for 6, and 31 for bringing it to A's turn to throw again, his expectation is in this caſe $\frac{31y}{36}$ which muſt be equal to x, hence $y=\frac{36x}{31}$, on the other hand A has 6 chances for 7, and 30 for x, that is, for giving B the dice, A's expectation is here $\frac{6+30x}{36}$, and muſt be equal to y, conſequently $x=\frac{31}{61}$, and the odds as 31 to 30.

Otherwiſe.

Let A's probability for throwing 7 at any aſſigned trial be x, that of the contrary y, u the probability of B's throwing 6 at any aſſigned trial, z for the contrary. From the nature of the play, the ſecond, fourth, ſixth, &c. throws belong to A, therefore $zx+zyzx+zyzyzx$, &c. or $zx+z^2xy+z^3y^2x$, &c. equal to $\frac{zx}{1-yz}$, by ſumming up the progreſſion,

greſſion, is A's total probability. The firſt, third, fifth, *&c.* throws appertain to B, therefore $u+zyu+zyzyu$, *&c.* or $u+zyu+z^2y^2u+z^3y^3u$, *&c.* equal to $\frac{u}{1-zy}$, is his total probability. For x, y, u, and z, ſubſtitute their reſpective values $\frac{6}{36}$, $\frac{30}{36}$, $\frac{5}{36}$, and $\frac{31}{36}$, the ſaid expreſſions become $\frac{31}{61}$, and $\frac{30}{61}$, whence the odds as 31 to 30, the ſame as before.

PROBLEM XXXII.

A and B play together with a pair of common dice, A wins if he throws 6, B wins if he throws 7. A takes one throw, then B two, then A two, and ſo on, by turns, until one of them wins; required the proportion of their chances.

SOLUTION.

Let p be the probability that A throws 6 the firſt throw, x that of his doing it in two throws, x' that of the contrary, y the probability that B brings up 7 in two throws, y' that of the contrary, then $\overline{1-p}\times y'\times x$ is the probability that A wins the game the ſecond time the dice come into his hands, and not before, $\overline{1-p}\times y'\times x'\times y'\times x$ is that of his ending

ing the play the third time the dice come into his hands, and not before, whence A's total probability is $p + \overline{1-p} \times y'x + \overline{1-p} \times y'^2 x'x + \overline{1-p} \times y'^3 \times x'x$, &c. $= p + \overline{1-p} \times x$ into $\overline{y' + y'^2 x' + y'^3 x'^2, \&c.}$ or by ſumming the progreſſion, equal to $p + \frac{\overline{1-p} \times x \times y'}{1 - y' \times x'}$. By the very ſame way of reaſoning B's probability for winning the game will be found equal to $\overline{1-p} \times y + \overline{1-p} \times y' \times x'y + \overline{1-p} \times y' \times x' \times y' \times x'y + \overline{1-p} \times y' \times x' \times y' \times x' \times y' \times x'y$, &c. equal to $\overline{1-p} \times y$ into $1 + y'x' + y'^2 x'^2 + y'^3 x'^3$, &c. or $\frac{\overline{1-p} \times y}{1 - y'x'}$, and the proportion of their chances as $p - y'x' + \overline{1-p} \times xy'$ to $\overline{1-p} \times y$, that is, as 10355 to 12276, by writing $\frac{5}{36}$, $\frac{355}{1296}$, $\frac{961}{1296}$, $\frac{396}{1296}$, $\frac{900}{1296}$, for p, x, x', y, y', reſpectively.

PROBLEM XXXIII.

Three perſons, A, B, C, throw in their turns a regular ball having four white faces and eight black ones, and he who ſhall firſt bring up a white one is to be reputed the winner; it is required to find the proportion of their reſpective probabilities for winning.

SOLUTION.

Let the probability for throwing a white face at any aſſigned trial be x, that for throwing a black one at any aſſigned throw y, then as the firſt, fourth, ſeventh, tenth, *&c.* throws are A's, $x+y^3x+y^6x+y^9x$, *&c.* is his probability for throwing a white face before either B or C. The ſecond, fifth, eighth, eleventh, *&c.* throws are made by B, therefore his probability is $yx+y^4x+y^7x+y^{10}x$, *&c.* Laſtly, becauſe the third, ſixth, ninth, twelfth, *&c.* throws are C's, it is evident that $y^2x+y^5x+y^8x+y^{11}x$, *&c.* is C's probability. The ſums of theſe geometrical progreſſions will give their reſpective probabilities of winning the game. But it is eaſily ſeen, that y^3 is the common ratio in each progreſſion, and x, xy, xy^2 their firſt terms, it follows that $1, y, y^2$, will expound the proportion of their chances that is in this problem $1, \frac{8}{12}, \frac{64}{144}$, or 9, 6, 4.

Corollary.

If there be any number of gameſters, A, B, C, D, E, *&c.* playing on the ſame conditions as before, their probabilities for winning the game will be proportional to $1, y, y^2, y^3, y^4$, *&c.*

PROBLEM XXXIV.

A, B, and C, out of a heap of twelve counters, whereof four are white and eight black, draw one counter at a time in this manner, A begins to draw firſt, B follows A, C follows B, then A begins again, and they continue to draw in the ſame order, till one of them, who is to be reputed the winner, draws the firſt white counter; what are their reſpective probabilities of winning the ſum depoſited, which ſuppoſe 1?

SOLUTION.

Firſt, it is evident that the play will be ended in nine drawings at moſt. Secondly, that the firſt, fourth, and ſeventh times of drawing, will (ſhould the play continue ſo long) belong to A, the ſecond, fifth, and eighth to B, the third, ſixth, and ninth to C. Thirdly, the probability of any one of the gameſters winning, when it is his turn to draw, is compounded of the probability of drawing a white counter at that time, and that of black ones being continually drawn before. The probability that A draws a white counter the firſt trial is $\frac{4}{12}$, but if this ſhould fail, then the probability that he ſucceeds when it is his turn to draw again, is $\frac{8}{12} \times \frac{7}{11} \times \frac{6}{10} \times \frac{4}{9}$, *viz.* that of three black ones being drawn ſucceſſively, and a white one next. After the ſame way of reaſoning we get $\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}\times\frac{4}{6}$, his probability of winning when he draws the laſt time, whence A's total expectation is $\frac{4}{12}+\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\times\frac{4}{9}+\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}\times\frac{4}{6}$. B's probability of winning the firſt trial is $\frac{8}{12}\times\frac{4}{11}$, that of his winning the ſecond is $\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}$, and $\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{4}{5}$, is his probability for winning the third time of drawing, and hence his total expectation is equal to $\frac{8}{12}\times\frac{4}{11}+\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}+\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{4}{5}$. Now as the third, ſixth, and ninth times of drawing belong to C, his total expectation will be expreſſed by $\frac{8}{12}\times\frac{7}{11}\times\frac{4}{10}+\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\times\frac{4}{7}+\frac{8}{12}\times\frac{7}{11}\times\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}\times\frac{3}{7}\times\frac{2}{6}\times\frac{1}{5}\times\frac{4}{4}$. Hence their reſpective probabilities are $\frac{231}{495}$, $\frac{159}{495}$, $\frac{105}{495}$, or in the proportion of 77, 53, 35.

The ſame univerſally.

Let *(n)* be the number of all the counters, *(a)* the number of white, and *(b)* the number of black ones, *(p)* the number of players A, B, C, D, *&c.* The probability that a white counter ſhall be taken out at any aſſigned drawing, and not before, is $\frac{b}{n}\times\frac{b-1}{n-1}\times\frac{b-2}{n-2}\,(r-1)\times\frac{a}{n-r+1}$, (*r* being put for the aſſigned number of drawings). Now if *r* be taken equal to 1, 2, 3, 4, *&c.* ſucceſſively; we ſhall have the following ſeries, $\frac{a}{n}+\frac{b}{n}\times\frac{b}{n-1}+\frac{a}{n}\times\frac{b-1}{n-2}\times\frac{b}{n-1}+\frac{a}{n}\times\frac{b}{b-1}\times\frac{b-1}{n-2}\times\frac{b-2}{n-3}$, &c. Put $P=\frac{a}{n}$, $Q=\frac{a}{n}\times\frac{b}{n-1}$, $R=\frac{a}{n}\times\frac{b}{n-1}\times\frac{b-1}{n-2}$, &c. and the ſeries becomes $\frac{a}{n}+\frac{b-1}{n-1}P+\frac{b-1}{n-2}Q+\frac{b-2}{n-3}R+\frac{b-3}{n-4}S$, &c. whence the law of continuation is manifeſt. This ſeries being continued to $(b+1)$ terms, will contain the ſum of the expectations of all the adventurers, becauſe the play muſt neceſſarily be ended in $b+1$ drawings at moſt, wherefore ſelecting from the ſaid ſeries the firſt, $1+p$, $1+2p$, &c. terms, their ſum will give A's expectation. The ſum of the ſecond, $2+p$, $2+2p$, $2+3p$, &c. terms, is B's expectation. Laſtly, the ſum of the third, $3+p$, $3+2p$, $3+3p$, $3+4p$, &c. terms, is C's, and proceeding in the ſame manner with the remaining gameſters, their expectations may be as eaſily determined.

If $a=$ unity, or there be but one white counter in the heap, the general ſeries becomes $\frac{1}{n}\times\overline{1+1+1+1+1}$, &c. becauſe $n=b+1$, whence it appears, that, if the whole number of counters be exactly diviſible by the number of players, and at the ſame time but one white counter in the whole, there will be no inequality in their ſeveral probabilities with regard to their ſituations in reſpect to the order of drawing.

If $a=4$, then $n=b+4$, whence the general theorem in this caſe becomes $1+\frac{n-4}{n-1}+\frac{\overline{n-4}\times\overline{n-5}}{\overline{n-1}\times\overline{n-2}}+\frac{n-4}{n-1}\times\frac{n-5}{n-2}\times\frac{n-6}{n-3}+\frac{n-5}{n-1}\times\frac{n-6}{n-2}\times\frac{n-7}{n-3}+\frac{n-6}{n-1}\times\frac{n-7}{n-2}\times\frac{n-8}{n-3}$, &c. the whole to be multiplied by $\frac{4}{n}$. If there are three players, A, B, and C, and $n=12$, the terms for their ſeveral expectations will be $1+\frac{8\times7\times6}{11\times10\times9}+\frac{5\times4\times3}{11\times10\times9}\times\frac{4}{12}$ for A, $\frac{8}{11}+\frac{7\times6\times5}{11\times10\times9}+\frac{3\times4\times2}{11\times10\times9}\times\frac{4}{12}$ for B, and $\frac{8\times7}{11\times10}+\frac{6\times5\times4}{11\times10\times9}+\frac{1\times2\times3}{11\times10\times9}\times\frac{4}{12}$ for C, which reduced to a common denominator will give the following proportions, *viz.*

For A.	*For* B.	*For* C.
11×10×9=990	10×9×8=720	9×8×7=504
8× 7×6=336	7×6×5=210	6×5×4=120
5× 4×3= 60	4×3×2= 24	3×2×1= 6
1386	954	630

Theſe ſums ſhould each be multiplied by $\frac{4}{12\times11\times10\times9}$ to give their reſpective probabilities. But as this fraction is common to them all, the multiplication may be omitted, and the proportion of their reſpective probabilities will be, as 1386, 954, 630, or dividing each by 18, as 77, 53, 35.

But when the products or terms of the ſeries are many, it will be found very tedious to ſum them up by common addition, wherefore to facilitate the operation it will be neceſſary to make uſe of ſome more expeditious method, ſuch as that given by Sir *Iſaac Newton* in his *Principia*, and very elegantly demonſtrated by Mr *Thomas Simpſon*, page 94 of his *Eſſays*, in the following manner.

Let $a, b, c, d, e, f, g, h,$ be the given ſeries, then by ſubtracting each of them from the next ſucceeding, there will be $-a+b$, $-b+c$, $-c+d$, $-d+e$, $-e+f$, $-f+g$, for the firſt differences. Again, taking each of theſe differences from its ſucceeding one, we have $a-2b+c$, $b-2c+d$, $c-2d+e$, $d-2e+f$, *&c.* for the ſecond differences, in the ſame manner the third differences will be $-a+3b$, $-3c+d$, $-b+3c$, $-3d+e$, $-c+3d$, $-3e+f$, *&c.* and the fourth $a-4b+6e-4d+e$, $b-4c+6d-4e+f$, *&c.* Let the firſt differences of the firſt order, *viz.* $-a+b$, be called D′, the firſt of the ſecond, *viz.* $a-2b+c$, D″, the firſt of the third order D‴, *&c.* then we ſhall have $a=a$, $b=a+D'$, $c=a+2b+D''$, $d=a-3b+3c+D'''$, *&c.* and from thence by ſubſtitution,

$$a=a$$
$$b=a+D'$$
$$c=a+2D'+D''$$
$$d=a+3D'+3D''+D'''$$
$$e=a+4D'+6D''+4D'''+D'''',$$

Where the law of continuation is manifeſt, the unciæ of the values of c, d, e, *&c.* being thoſe of a binomial, raiſed to the ſecond, third, fourth powers, *&c.* Therefore if n be put for the number of terms in the propoſed ſeries, $a+b+c+d$, *&c.* whoſe ſum we are about to find, the value of the next term in the progreſſion after the laſt in that ſeries, or that whoſe place is defined by $n+1$, will, it is plain, be equal to $a+nD'+n\times\frac{n-1}{2}D''+n\times\frac{n-1}{2}\times\frac{n-2}{3}D'''+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times D''''$, *&c.* And ſo much will the ſaid ſum be increaſed by augmenting n, the given number of terms, by unity; hence, by the method of increments, page 87 of the ſame treatiſe, we get $a+b+c+d+e$, *&c.* equal to $na+n\times\frac{n-1}{2}D'+n\times\frac{n-1}{2}\times\frac{n-2}{3}D''+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}D'''+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}D''''$.

Let us now ſuppoſe that three perſons, A, B, C, playing together on the ſame conditions as in *Problem* XXXIV, the number of counters were 100 inſtead of 12, ſtill preſerving the ſame number 4 of white counters; and that it were required to determine the expectations of A, B, and C. Now becauſe here are 100 counters, if we divide 97 by 3, we ſhall find that 33 terms of the ſeries will enter the computation of A, 32 into that of B, and 32 into the computation of C, for the play muſt evidently be ended in 97 drawings at moſt. Now if from the general ſeries $\frac{a}{n} + \frac{b-1}{n-1} P + \frac{b-1}{n-2} Q + \frac{b-2}{n-3} R + \frac{b-3}{n-4} S$, &c. we ſelect the 1, 4, 7, &c. terms, until their number becomes 33, the numerator of the ſum of thoſe terms, when reduced to a common denominator, will be proportional to A's expectation, which, by writing 4 for *a*, and 100 for *n*, is equal to $99 \times 98 \times 97 + 96 \times 95 \times 94 + 93 \times 92 \times 91 + 90 \times 89 \times 88$, &c. continued until the products are in number 33, wherefore the laſt product will be $1 \times 2 \times 3$, hence we get, by inverting the terms,

A	D'	D''	D'''	
$3 \times 2 \times 1 = 6$				
$6 \times 5 \times 4 = 120$	114	270		
$9 \times 8 \times 7 = 504$	384	432	162	$D'''' = 0$
$12 \times 11 \times 10 = 1320$	816	594	162	
$15 \times 14 \times 13 = 2730$	1410			
&c.				

Now to find the ſum of 6 + 120 + 504 + 1320 + 2730, &c. continued to 33 terms. Here we have $a = 6$, $n = 33$, $D' = 114$, $D'' = 270$, $D''' = 162$, $D'''' = 0$, whence the ſum of the terms, by the above theorem, *viz.* $na + n \times \frac{n-1}{2} D' + n \times \frac{n-1}{2} \times \frac{n-2}{3} D'' + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} D'''$, &c. is equal to 8162550. By proceeding in the ſame manner with the ſecond, fifth, eighth, &c. terms, their ſum (neglecting the common denominator) will be found equal to 7839216, B's chances for winning. The third, ſixth, ninth, &c. terms, being ordered in the like manner, will give C's chances equal to 7525584, or in lower terms, by dividing each by 198, in the ratio 41225, 39592, 38008.

If the number of all the counters were 500, and the number of white ſtill 4, then the number of terms repreſenting the chances of A, B, and C, would be 497, which divided by 3 leaves 2 remaining, which ſhews the chances of A and B do each contain 166 terms, C's 165 only. But from what has been ſaid in the foregoing examples it will appear, that the firſt or greateſt products of the ſeveral players A, B, C, will be 499×498 ×497×496×495×494×493×492×491, reſpectively, whence it follows, that the laſt or leaſt products will be $4 \times 3 \times 2$, $3 \times 2 \times 1$, $5 \times 4 \times 3$, for A, B, and C.

PROBLEM XXXV.

To find what probability there is that in taking at random ſeven counters, out of a heap conſiſting of four white and eight black, three of the ſeven ſhall be white ones.

SOLUTION.

Imagine the ſeven counters to be drawn one at a time. The probability of drawing a white one the firſt time is $\frac{4}{12}$. If a white one be ſo drawn, then as there remains 11 counters in all, *viz.* 3 white and 8 black, the probability of drawing a white one next will be $\frac{3}{11}$, hence $\frac{4}{12} \times \frac{3}{11}$ is the probability of drawing two white ones ſucceſſively, and in like manner that of taking out three white counters ſucceſſively is $\frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$. Now if three white ones are already drawn, there will remain only one white counter, whence the probability of drawing 4 black counters, will by the ſame way of reaſoning appear to be $\frac{8}{9} \times \frac{7}{8} \times \frac{6}{7} \times \frac{5}{6}$, and conſequently that of drawing three white counters and four black ones from the ſaid heap, is $\frac{2}{10} \times \frac{3}{11} \times \frac{4}{12} \times \frac{5}{6} \times \frac{6}{7} \times \frac{7}{8} \times \frac{8}{9} = \frac{1}{99}$. But as there is the very ſame chance for them to come up in any other aſſigned order than is here ſpecified, it follows, that as many different ways as three things can be taken or combined in 7, (that is 35, by *Problem* II.) juſt ſo often ſhould the aforeſaid probability be repeated to give the required one, whence $\frac{1}{99} \times 35$ is the probability of drawing preciſely 3 white counters and 4 black ones, to which add that of drawing 4 white counters and 3 black ones, *viz.* $\frac{56}{792}$, gives $\frac{14}{33}$, the true probability required, becauſe it is generally underſtood that according to the law of play, he that does more than is required of him to be a winner, is not deemed a loſer, unleſs it be expreſſly agreed upon to be ſo.

Otherwiſe.

Conceive the 12 counters parted into two different heaps, one containing 4 white counters, and the other 8 black ones. The combinations of three in four are $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} = 4$, and thoſe of four in eight $\frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} \times \frac{5}{4} = 70$, therefore 4×70, or 280, are the different ways for taking 3 white counters and 4 black ones, from a heap conſiſting of 4 white and 8 black. The whole number of different ways whereby may be taken 7 counters out of the ſaid heap, being $\frac{12.11.10.9.8.7.6}{1.2.3.4.5.6.7}=792$, it follows, that $\frac{280}{792}$, is the probability of drawing three white counters and not more, to which add that of taking 4 white and 3 black, *viz.* $\frac{56}{792}$, the ſum $\frac{14}{33}$ is the required probability, the ſame as before.

Corollary I.

Let (a) be the number of white counters, (b) the number of black ones, (n) the ſum of a and b, (c) the number of counters to be drawn from the heap, (p) the number of white counters to be found preciſely in c, $m=c-p$. Now ſuppoſe the whole number of counters placed in two heaps, one containing all the white counters, the other all the black ones, the combinations of p things in (a) being $\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}(p)$, and thoſe of m in b, $\frac{b}{1}\times\frac{b-1}{2}\times\frac{b-2}{3}\times\frac{b-3}{4}(m)$, it follows, that $\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}(p)\times\frac{b}{1}\times\frac{b-1}{2}\times\frac{b-2}{3}\times\frac{b-3}{4}(m)$ will expreſs the different ways by which p white counters, and m black ones, can be drawn, this being divided by $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$ (c), being all the different ways that (c) things can be taken in the number (n) gives the probability required, equal to —— ,

$$\frac{\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}\,(p)\text{ into }\frac{b}{1}\times\frac{b-1}{2}\times\frac{b-2}{3}\times\frac{b-3}{4}\,(m)}{\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}\times\frac{n-5}{6}\,(c)}$$

or by actual division equal to

$$\frac{\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}\,(p)\text{ into }b\times\overline{b-1}\times\overline{b-2}\times\overline{b-3}\,(m)\text{ into }c\times\overline{c-1}\times\overline{c-2}\times\overline{c-3}\,(c-m)}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\overline{n-6}\,..\,(c)}$$

Now to contract this expreſſion as much as poſſible, let the denominator be parted into two factors $n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\,(a)$, and $b\times\overline{b-1}\times\overline{b-2}\times\overline{b-3}\,(c-a)$ which may be done becauſe $a+b=n$, whence the expreſſion becomes

$$\frac{\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}\,(p)\text{ into }c\times\overline{c-1}\times\overline{c-2}\times\overline{c-3}\,(c-m)}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\,(a)}\text{ into }\frac{b\times\overline{b-1}\times\overline{b-2}\times\overline{b-3}\times\overline{b-4}\,(m)}{b\times\overline{b-1}\times\overline{b-2}\times\overline{b-3}\,(c-a)}$$

but this laſt factor is equal to $\overline{b+a-c}\times\overline{b+a-c-1}\times\overline{b+a-c-2}\times\overline{b\times a-c-3}\,(m-c+a)$ therefore the required probability by this contraction becomes ——

$$\frac{\frac{a}{1}\times\frac{a-1}{2}\times\frac{a-2}{3}\times\frac{a-3}{4}\times\frac{a-4}{5}\,(p)\text{ into }c\times\overline{c-1}\times\overline{c-2}\times\overline{c-3}\,(p)\text{ into }\overline{n-c}\times\overline{n-c-1}\times\overline{n-c-2}\times\overline{n-c-3}\,(a+p)}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times\overline{n-5}\times\overline{n-6}\times\overline{n-7}\,(a)}$$

by writing p, n, and $a-p$, for their equals

$c-m$, $a+b$ and $m+a-c$, reſpectively. If $p=0$, then $c=m$, and $\frac{b}{n}\times\frac{b-1}{n-1}\times\frac{b-2}{n-2}\,(c)$, or $\frac{n-c}{n}\times\frac{n-c-1}{n-1}\times\frac{n-c-2}{n-2}\,(a)$ is the probability for taking preciſely (c) black counters.

Corollary II.

The probability of taking c, or a greater number of black counters, is equal to the ſum of the ſeveral probabilities for taking preciſely c, 1.2.3, &c. $(c-p)$ black ones, ſubtracted from the probability of not taking $c-1$ black ones preciſely.

Example I.

There is a lottery conſiſting of 1000 tickets, among which are three particular prizes, what is the probability that a perſon in taking 30 tickets ſhall have all the three prizes? In this caſe it matters not whether there are any other prizes in the lottery beſides thoſe three particular ones, therefore the remaining 997 tickets may be all conſidered as blanks, hence $a=3$, $b=997$, $n=1000$, $c=30$, $p=3$, and the probability ſought equal to $\frac{\frac{3}{1}\times\frac{2}{2}\times\frac{1}{3}\text{ into }30\times29\times28}{1000\times999\times998}=\frac{207}{8308350}=\frac{1}{40927}$ nearly.

Example II.

The ſame thing being ſuppoſed as in the laſt example. Let it be required to find the probability of taking one or more prizes. Here $a = 3$, $b = 997$, $n = 1000$, $c = 30$, as before, but $p = 0$, then $\frac{970 \times 969 \times 968}{1000 \times 999 \times 998} = \frac{90985224}{99700200}$ is the probability of drawing all blanks, which ſubtracted from 1 leaves $\frac{8714976}{99700200}$ for that of not drawing all blanks, or the probability of drawing one or more prizes.

Example III.

In a lottery conſiſting of 40000 tickets, among which there are three particular prizes, how many tickets muſt be taken to make it an equal chance that one or more of thoſe prizes ſhall be drawn? Here $n = 40000$, $a = 3$, $b = 39997$, $p = 0$, whence becauſe (n) is a very large number, the expreſſion $\frac{n-c}{n} \times \frac{n-c-1}{n-1} \times \frac{n-c-2}{n-2}$ (a) becomes $\frac{n-c}{n} \times \frac{n-c}{n} \times \frac{n-c}{n}$ (a) very nearly, equal to $\overline{\frac{n-c}{n}}\Big|^{3}$ which muſt be $\frac{1}{2}$, therefore $c = \sqrt[a]{\frac{n}{2}} = 8252$.

Example IV.

In a lottery conſiſting of 100000 tickets, in which there are 10000 prizes, and 90000 blanks, how many tickets ſhould be purchaſed to have an equal chance for one or more prizes? By *Corol.* I. $\frac{b}{n} \times \frac{b-1}{n-1} \times \frac{b-2}{n-2} \times \frac{b-3}{n-3}$

(c) is the probability of drawing c tickets all blanks, wherefore becauſe n and b are in this caſe very large numbers, we may reject the numerical quantities as inconſiderable, which done, it becomes $\frac{b^c}{n^c}$, this ſubtracted from 1 leaves the probability of drawing one or more prizes in the required number of tickets, hence $1 - \frac{b^c}{n^c} = \frac{1}{2}$, and conſequently $c =$ $\frac{Log.\,2}{Log.\,n - Log\,b} = \frac{.30103}{5,00000 - 495424} = 6,6$ nearly, which ſhews that in taking 7 tickets the adventurer has ſomething more than an equal chance for one or more prizes, but in taking 6, ſomething leſs.

It may be neceſſary here to obſerve, that as the expreſſions $\frac{b}{n} \times \frac{b-1}{n-1} \times \frac{b-2}{n-2} \times \frac{b-3}{n-3}$ (c) $\frac{n-c}{n} \times \frac{n-c-1}{n-1} \times \frac{n-c-2}{n-2}$ (a) are equal to each other, as is evident by *Corollary* I. the reader may conclude that either of them would have ſerved for the ſolution of the two laſt examples; but notwithſtanding they are alike, yet when the numerical quantities in each are rejected, they differ very much, the former then becoming $\frac{b^c}{n^c}$, which is a theorem relating to throwing of dice, as may be ſeen by the rules given in the introduction, and conſequently ſuppoſes the ratio of a to b, continues invariable. — Wherefore in all calculations of chances relative to lotteries, the above theorem will prove defective, and more eſpecially in thoſe caſes wherein the ratio of the blanks to the prizes may admit of great alteration, as in the third example, where, there being but three prizes in all, that ratio might very probably change from $\frac{3}{40000}$ to that of equality, during the drawing of the lottery. But when the proportion between the blanks and prizes is often repeated, as indeed it uſually is in lotteries, problems of this ſort may be ſolved ſufficiently near the truth by conſidering them as thoſe depending on the caſting of dice.

PROBLEM XXXVI.

What is the probability that in taking at random *(m)* counters from a heap conſiſting of *(a)* white counters, *(b)* black ones, and *(c)* red ones, there ſhall ſtill come out preciſely *(r)* of the firſt colour, *(s)* of the ſecond, and *(t)* of the third?

SOLUTION.

Conceive the counters diſtributed into three heaps, one heap containing all the red ones, another all the black ones, and the third all the white ones. Now find the ſeveral combinations of (r) things in (a), thoſe of (s) things in (b), and of (t) in (c), which are reſpectively $\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3}\ (r)$, $\frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} \times \frac{b-3}{4}\ (s)$, and $\frac{c}{1} \times \frac{c-1}{2} \times \frac{c-2}{3} \times \frac{c-3}{4} (t)$, conſequently $\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3}\ (r)$ into $\frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} \times \frac{b-3}{4}\ (s)$ into $\frac{c}{1} \times \frac{c-1}{2} \times \frac{c-2}{3} \times \frac{c-3}{4}\ (t)$ are all the different ways by which the aſſigned counters can be taken, which being divided by the combinations of (m) things in (n), *viz.* $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}\ (m)$ gives the probability required, equal

$$\text{to } \frac{\frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3}\ (r) \text{ into } \frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} \times \frac{b-3}{4} (s)}{\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6} \times \frac{n-6}{7} \times}$$

$$\text{into } \frac{\frac{c}{1} \times \frac{c-1}{2} \times \frac{c-2}{3} \times \frac{c-3}{4}\ (t)}{\frac{n-7}{8}\ (m)}$$

Note, This method of ſolution extends to any other caſe of the problem, as when there are four or a greater number of colours in the propoſed heap.

Example I.

A wagers with B that from a heap of 30 cards, *viz.* 10 of each ſort, hearts, ſpades, and clubs, he will at a venture draw three, ſo that there ſhall be one of each; required his probability of winning? Here a, b, and c, are each $= 10$, and r, s, and t, each $= 1$, $m = 3$, $n = 30$, and the probability ſought equal to $\frac{10\times10\times10}{\frac{30}{1}\times\frac{29}{2}\times\frac{28}{3}} = \frac{100}{406}$.

Example II.

In a heap of cards containing 10 in number, wherein there are 5 diamonds, and 5 clubs; what is the probability that in drawing 5 of them at random, there ſhall come out two or more diamonds? Here $n = 10$, $m = 5$, $a = 5$, $b = 5$, $r = 2$, $s = 3$, whence we get $\frac{\frac{5}{1}\times\frac{5}{1}\times\frac{4}{2}\times\frac{3}{3}\times\frac{2}{1}}{\frac{10}{1}\times\frac{9}{2}\times\frac{8}{3}\times\frac{7}{4}\times\frac{6}{5}}$, that is, $\frac{25}{252}$, the probability of taking one diamond preciſely, which ſubtracted from 1 leaves $\frac{227}{252}$ for the contrary, *viz.* for having either more than one or elſe none, but $\frac{1}{252}$ is the probability for taking all clubs, therefore $\frac{227}{252} - \frac{1}{252}$ or $\frac{226}{252}$ is the probability of drawing two or more diamonds.

PROBLEM XXXVII.

The ſame things being ſuppoſed as in the laſt problem, to find the probability that in drawing the counters ſingly, the (p) firſt of them ſhall come out of the firſt ſort, and the next after of the ſecond ſort.

SOLUTION.

The number of the firſt ſort being (a), of the ſecond ſort (b), the whole number of counters (n), the probability of taking one of that ſort whoſe number is (a), will be $\frac{a}{n}$, if this be ſuppoſed to have happened, there remains $a-1$ of that ſort, and therefore the probability of taking one of thoſe next is $\frac{a-1}{n-1}$, that of ſucceeding twice ſucceſſively is $\frac{a}{n} \times \frac{a-1}{n-1}$, and conſequently the probability of taking all the firſt (p) counters of the firſt ſort is $\frac{a}{n} \times \frac{a-1}{n-1} \times \frac{a-2}{n-2} \times \frac{a-3}{n-3}$, &c. (p) Now if theſe ſhould happen to be ſo drawn, it will follow, becauſe the probability of taking one of the ſecond ſort next is $\frac{b}{n-p}$, that $\frac{b}{n-p}$ multiplied by $\frac{a}{n} \times \frac{a-1}{n-1} \times \frac{a-2}{n-2} \times \frac{a-3}{n-3}$, &c. (p) or $\frac{b}{n} \times \frac{a}{n-1} \times \frac{a-1}{n-2} \times \frac{a-2}{n-3} \times \frac{a-3}{n-4}$ $(p+1)$ is the probability required.

PROBLEM XXXVIII.

The chances for happening or failing of an event in any one trial, being equal, in how many trials may I undertake upon equality of chance, that the propoſed event ſhall happen (r) times or oftener?

SOLU-

SOLUTION.

Let *(a)* be the chances for happening or failing at each trial, *(x)* the number ſought. By *Prob.* VIII. the probability of happening *(r)* times, or oftener in x trials, is equal to

$$\frac{a^x + x \times a^x + x \times \frac{x-1}{2} a^x + x \times \frac{x-1}{2} \times \frac{x-2}{3} a^x \; (r)}{2a^x}$$

which muſt be $\frac{1}{2}$. Wherefore ſince one half of the number of terms of the binomiol $\overline{a+a}^x$ muſt be equal to the other, it is evident that $x = 2r - 1$, becauſe the whole number of terms of any power of a binomial is equal to the exponent of that power plus 1.

Example I.

In how many throws with a regular ball having 13 red, and 13 black faces, may I undertake to make it as likely that 4 red faces ſhall appear as 4 black ones? Here $r = 4$, whence $x = 7$, that is in 7 throws it is juſt as probable that 4 red faces ſhall come up, as not.

Example II.

In a pack of 26 cards, 13 of which are black, and 13 red, if 11 cards be dealt, how many is there an equal chance of being red ones? Here is given $x = 11$, whence $r = 6$.

Note, In this example the ſolution is not quite exact, becauſe the number of cards in the pack diminiſhes after each trial, whereas in the other, the number of faces upon the ball continue the ſame after every trial.

PROBLEM XXXIX.

There being *(a)* chances for an event happening, and *(b)* chances for its failing; in how many trials may one undertake that the ſaid event, upon equality of chance, ſhall happen once or oftener.

SOLUTION.

Let x be the number of trials ſought, by *Problem* VIII, $\frac{b^x}{a+b^x}$ is the probability that the propoſed event does not happen at all in thoſe trials, which by the nature of the problem muſt be equal to $\frac{1}{2}$, whence $2 \times b^x = \overline{a+b}^x$, and by taking the logarithm of the equation, we get $x \times Log.\, b + Log.\, 2 = x \times Log.\, \overline{a+b}$, $x = \frac{Log.\, 2}{Log.\, n}$ by writing n for $\frac{a+b}{b}$. If in the equation $\overline{a+b}^x = 2 \times b^x$, we write p for $\frac{a}{b}$, it will be changed into $\overline{1+p}^x = 2$, which in Logarithms is $x \times Log.\, \overline{p+1} = Log.\, 2$. Now if $a=b$, or $p=1$, x will be $=\frac{Log.2}{Log.2}=1$. But if p be ſmall in compariſon with unity, let the logarithm of $1+p$ be expreſſed by a ſeries $p-\frac{1}{2}p^2+\frac{1}{3}p^3-\frac{1}{4}p^4+\frac{1}{5}p^5$, &c. by taking only the firſt term we get $px=Log.2.=0{,}693$, or $px=.7$, nearly, and $x=\frac{0.7}{p}$, we had before $px=1$, in which caſe $x=\frac{1}{p}$, from theſe two values of x, we may infer, that in all caſes it may be taken equal to $\frac{0.7}{p}+0{,}3$, be the value of (p) what it will.

Example I.

In how many throws with a ſingle die may one undertake to throw either the ace or duce, upon equality of chance? Here $a=2$, $b=4$, $p=\frac{1}{2}$, whence $x=1{,}7$.

Example II.

In how many throws with a pair of common dice may one undertake to bring up both aces? There being 36 chances upon two dice, and only one chance for bringing up the two aces at any one throw, it follows, that $a=1$, $b=35$, $p=\frac{1}{35}$, hence $x=24{,}8$; which ſhews that in 25 throws it is ſomething more than an equal chance, that the two aces ſhall come up together.

Example III.

Suppoſe a lottery conſiſting of a very great number of tickets, wherein the blanks are to the prizes as 39 to 1; to find how many tickets muſt be taken to make it an equal chance for one or more prizes? Here $a=1$, $b=39$, $p=\frac{1}{39}$, whence $x=27,6$.

Note, The ſolution of this example is not ſo exact as the ſolution of the others, becauſe the number of tickets drawn diminiſh the number of thoſe remaining in the wheel; whereas in throwing with dice the caſe is otherwiſe, the ratio of a to b, continuing the ſame, which in drawing of tickets varies, and is greater or leſs according to the number of blanks that have been drawn, yet when the number of tickets in the lottery is very great, the alteration of the ratio $\frac{a}{b}$ being then very ſmall, this method of ſolution will be ſufficiently near the truth. Theſe examples might have accurately been ſolved by the firſt theorem, *viz.* $x=\frac{\text{Log. }2}{\text{Log. }\overline{a+b}-\text{Log. }b}$, but the latter method being much more expeditious, may perhaps be thought preferable.

PROBLEM XL.

Things remaining as in the laſt problem; it is required to find the number of trials neceſſary to make it an equal chance that the event ſhall happen twice or oftener.

SOLUTION.

It appears by *Prob.* VIII, that $\frac{b^x + xb^{x-1}a}{\overline{a+b}^x}$ is the probability that the event does not happen twice or oftener in x trials, therefore $b^x + xb^{x-1}a = \frac{1}{2} \times \overline{a+b}^x$ or $1+px = \frac{\overline{1+p}^x}{2}$, and *Log.* $\overline{1+px} = x \times$ *Log.* $\overline{1+p}$ — *Log.* 2. But as theſe logarithms muſt be expounded by ſeries, it will, in order to make thoſe ſeries converge ſwiftly, be convenient to ſubſtitute $2+v$ for px, whence *Log.* $\overline{v+3} = \frac{2+v}{p} \times$ *Log.* $\overline{1+p}$ — *Log.* 2. Now the logarithm of m being given, that of $m+z$ is equal to $\frac{2z}{2m+z}$ multiplied by $1 - \frac{z}{12} \times \overline{\frac{1}{m+z} - \frac{1}{m}} + \frac{z^3}{360} \times \overline{\frac{1}{\overline{m+z}^3} - \frac{1}{m^3}} - \frac{z^5}{1260} \times \overline{\frac{1}{\overline{m+z}^5} - \frac{1}{m^5}}$, &c. + *Log.* m. For the inveſtigation of this uſeful ſeries ſee Mr *Mac Laurin*'s *Fluxions*, page 679, as this ſeries converges very ſwift, it will be

ſuffi-

$1+px+\frac{p^2x^2}{2}+\frac{p^3x^3}{2.3}+\frac{p^4x^4}{2.3.4}$, &c. $(r)=\overline{\frac{1+p}{2}}\Big|^{x}$, for px put $r+v$, we have $1+r+v+\frac{\overline{r+v}^2}{2}+\frac{\overline{r+v}^3}{2.3}+\frac{\overline{r+v}^4}{2.3.4}(r)=\overline{\frac{1+p}{2}}\Big|^{\frac{r+v}{p}}$, or by expanding the ſeveral terms, $1+r+v+\frac{r^2+2rv+v^2}{2}+\frac{r^3+3r^2v+3rv^2+v^3}{2.3}$, &c. $(r)=\overline{\frac{1+p}{2}}\Big|^{\frac{r+v}{p}}$,

That is,

$$\left.\begin{array}{l}1, \text{ into } 1+r+\frac{r^2}{2}+\frac{r^3}{2.3}, \text{ \&c. } (r)\\ v, \text{ into } 1+r+\frac{r^2}{2}+\frac{r^3}{2.3}, \text{ \&c. } (r-1)\\ \frac{v^2}{2}, \text{ into } 1+r+\frac{r^2}{2}+\frac{r^3}{2.3}, \text{ \&c. } (r-2)\\ \frac{v^3}{2.3}, \text{ into } 1+r+\frac{r^2}{2}+\frac{r^3}{2.3}, \text{ \&c. } (r-3)\\ \text{\&c.} \qquad \text{\&c.}\end{array}\right\}=\overline{\frac{1+p}{2}}\Big|^{\frac{r+v}{p}}$$

Put $1+r+\frac{r^2}{2}+\frac{r^3}{2.3}+\frac{r^4}{2.3.4}(r)=d$; $1+r+\frac{r^2}{2}+\frac{r^3}{2.3}(r-1)=e$, $1+r+\frac{r^2}{2}(r-2)=f$, then $d+e\dot{v}+\frac{fv^2}{2}+\frac{gv^3}{2.3}+\frac{hv^4}{2.3.4}$, &c. $=\left.\frac{\overline{1+p}}{2}\right|^{\frac{r+v}{p}}$, and conſequently the logarithm of $2d+2ev+\frac{fv^2}{1}+\frac{gv^3}{3}+\frac{hv^4}{3.4}$, &c. $=\frac{r+v}{p}\times$ *Log.* $\overline{1+p}$, the logarithm of $\overline{1+p}$ is $p-\frac{p^2}{2}+\frac{p^3}{3}-\frac{p^4}{4}$, &c. but the logarithm of the other ſide of the equation not being ſo eaſily obtained, it will be proper to ſhew the inveſtigation of it. The fluxion of the logarithm of any quantity is equal to the fluxion of that quantity divided by the quantity itſelf, whence $\frac{2e\dot{v}+2fv\dot{v}+gvv\dot{v}}{2d+2ev+fv^2+\frac{gv^3}{3}}$, &c. is the fluxion of the logarithm required. Now to find the fluent, firſt divide $2e+2fv+gv^2$, &c. by $2d+2ev+fv^2+\frac{gv^3}{3}$, &c. as in common arithmetic. Secondly, multiply the quotient, *viz.* $\frac{e}{d}+\frac{fv}{d}+\frac{e^2v}{d^2}+\frac{gv^2}{2d}+\frac{3efv^2}{2d^2}+\frac{e^3v^2}{d^3}$, &c. by $\dot{v}$. Thirdly, find the fluent of each term. And laſtly, correct the fluent by adding the logarithm of $2d$, (for v being made equal to nothing, the whole vaniſhes) whence the true fluent is *Log.* $2d$ plus $\frac{ev}{d}+\frac{fv^2}{2d}-\frac{eevv}{2dd}+\frac{gv^3}{6d}-\frac{efv^3}{2d^2}+\frac{e^3v^3}{3d^3}$, &c. equal to $r+v$ into $1-\frac{p}{2}+\frac{p^2}{3}-\frac{p^3}{4}$, &c. equal to $r+v$ nearly, p being ſuppoſed very ſmall, by proper reduction we get $v+\frac{fd-ee}{2d\times\overline{e-d}}\times v^2$ $+\frac{gd-3ef+\frac{2e^3}{d}}{6d\times\overline{e-d}}\times v^3$, &c. $=\frac{r-Log.\,2d}{e-d}\times d$, for this laſt put z, and for the co-efficients of v, v^2, v^3, &c. put 1, B, C, D, &c. whence by reverting the ſeries we have $v=z-Bz^2+\overline{2B^2-C}\times z^3$, &c. therefore $px=r+v$ is equal to $r+z-Bz^2+\overline{2B^2-C}\times z^3$, or becauſe the ſeries converges very ſwift, $px=r+z$ nearly, that is, equal to $\frac{re-Log.\,2d}{e-d}\times d$, by reſtoring the value of z. Now if r be taken equal to 1, 2, 3, 4, 5, &c. ſucceſſively, px becomes ,69314, 1,6784, 2,6743, 3,6720, 4,6709, and 5,6702 reſpectively, conſequently, from hence it appears that the difference between any of the correſponding values of r and px, is very near conſtantly the ſame, *viz.* $\frac{3}{10}$, and that when p is very ſmall, $x=\frac{r-,3}{p}$, but when $a=b$, x is then $=\frac{2r-1}{p}$ by

Problem XXXVIII. Therefore as the value of $p\dot{x}$ is ever between $r-,3$ and $2r-1$ it follows, that x may in all caſes be taken equal to $\frac{b}{a}\times\overline{r-,3}+r-.7$.

Note, The hyperbolic logarithms are uſed both in this and the foregoing problem.

Example I.

In how many throws with three common dice may one undertake to throw the aces? The chances for failing at any aſſigned throw are 215, for happening only 1, whence $a=1$, $b=215$, $x=\frac{b}{a}\times\overline{r-,3}+r-,7=150$, r being in this caſe $=1$.

Example II.

In a lottery conſiſting of a very great number of tickets, wherein the blanks are to the prizes as 9 to 1, how many muſt be taken to make it an equal chance that 4 of them at leaſt ſhall be prizes? Here $a=1$, $b=9$, $r=4$, whence $x=36$, the number required.

Example III.

Suppoſe a lottery like the foregoing, wherein there are 39 blanks to one prize, how many tickets muſt be purchaſed that the buyer may have an equal chance for one or more prizes, $a=1$, $b=39$, $r=1$, and $x=27$.

Example IV.

In a pack of 52 cards, conſiſting of 13 of each ſuit, if 13 cards be dealt to me, how many may I on equality of chance expect to be trumps? In this caſe there is given $a=13$, $b=39$, and $x=13$ to find r, which by the equation above will be found equal to 4 nearly; therefore in the game of whiſt it is ſomething leſs than an equal chance for any particular player (the dealer excepted) having 4 trumps.

Example V.

A perſon playing with a ſingle die, determines to caſt it (m) times, how many times may he undertake, upon equality of chance, to throw an ace? Here $a=1$, $b=5$, $x=m$, therefore $r=\frac{m+2.2}{6}$.

PROBLEM XLII.

Any number of chances being propoſed, to find the probability of their being produced in any aſſigned order, without any limitation of the number of trials in which they may happen to be produced.

SOLUTION.

Let the propoſed chances be a, b, c, d, e, f, and this the order in which they are to be produced. The probability of producing *a* firſt is $(n = a+b+c+d+e+f)\ \frac{a}{n}$, that of producing (*a*) firſt and (*b*) next is $\frac{a}{n} \times \frac{b}{n-a}$, the probability of producing *a*, *b*, *c*, before any of the reſt, but in the aſſigned order, is $\frac{a}{n} \times \frac{b}{n-a} \times \frac{c}{n-a-b}$, and by the ſame ſort of proceſs we get the probability of producing *a*, *b*, *c*, *d*, *e*, before *f*, and in the aſſigned order, equal to

$$\frac{a \times b \times c \times d \times e}{n \times \overline{n-a} \times \overline{n-a-b} \times \overline{n-a-b-c} \times \overline{n-a-b-c-d}}$$

whence the law of continuation is manifeſt. Let it be required by way of example, to find the probability of throwing, with a pair of common dice, the chances IV, V, VI, VIII, IX, X, before VII, without any reſtriction as to the order in which thoſe chances may happen to ariſe, but only that they may all come up before VII. Let the propoſed chances, or rather thoſe for their happening, *viz.* 3, 4, 5, 5, 4, 3, and 6, be called *a*, *b*, *c*, *d*, *e*, and *m*, reſpectively. The probability of throwing them in an aſſigned order is

$$\frac{a + b \times c \times}{n \times \overline{n-a} \times \overline{n-a-b} \times \overline{n-a-b-c} \times \overline{n-a-b-c-d} \times}$$

$$\frac{d \times e \times f}{\overline{n-a-b-c-a-e} \times \overline{n-a-b-c-d-e-f}}.$$

But as that is not the thing particularly here required, only that the chances (*m*) may come up laſt, there will be as many different parts like this, as there are changes to be made with ſix letters, which being 720, we ſhall have ſo many parts like the above to be added together, in order to have a compleat ſolution to the problem. But the chances IV and X, V and IX, VI and VIII, being reſpectively the ſame, thoſe 720 will be reduced to 90, which added together, and the ſum multiplied by 8, will give the probability required; yet notwithſtanding this reduction the operation would be ſtill very laborious, if not altogether impracticable, for which reaſon it will be neceſſary to have an approximation, by ſuppoſing all the chances except that which is to happen laſt, equal to ſome mean chance which may be found by ſeeking the ſeveral probabilities of throwing all the propoſed chances before VII, and taking $\frac{1}{6}$ of their ſum for the mean probability, whence the mean chance is known. In the problem before us, the ſeveral probabilities of throwing all the other chances are $\frac{1}{9}$, $\frac{4}{10}$, $\frac{5}{11}$, $\frac{5}{11}$, $\frac{4}{10}$, and $\frac{1}{9}$, their ſum is $\frac{392}{165}$, $\frac{1}{6}$ of it is $\frac{59}{149}$ nearly, the mean probability. Let z repreſent the correſponding mean chance, then $\frac{z}{z+6}=\frac{59}{149}$, and $z=3\frac{14}{15}$, now that expreſſion which before denoted the probability of their being thrown in an aſſigned order becomes $\frac{z\times z\times z\times z\times z\times z}{\overline{6z+m}\times\overline{5z+m}\times\overline{4z+m}\times\overline{3z+m}\times\overline{2z+m}\times\overline{z+m}}$, and there being no difference in their probabilities for coming up in different orders when they are thus reduced to a mean chance, it follows that the laſt expreſſion multiplied by 720 gives the probability ſought, equal to $\frac{65798}{10000000}$. But if it was farther required not only to throw all the other propoſed chances before VII, but alſo to do it in a certain number of trials, the problem might eaſily be ſolved by imagining a mean chance, becauſe without which we ſhould have oftentimes as many different expreſſions to add up as there are changes in the propoſed chances that are to be firſt produced, whereas by this method of a mean chance they are reduced to only one certain quantity, that being multiplied by that number of changes will give the probability required.

PROBLEM XLIII.

To find the number of chances for throwing preciſely any given number of points *(p)* in one throw with four dice, each having *(p)* or a greater number of faces.

SOLUTION.

In order to facilitate the ſolution of this and the following problem, I ſhall lay down a lemma which was communicated to me by my ingenious friend Mr *William Payne*, teacher of mathematics.

The Lemma.

The ſum of 1, 3, 6, 10, 15, 21, 28, 36, &c. continued to *(n)* number of terms is equal to $\frac{n+2}{1}\times\frac{n+1}{2}\times\frac{n}{3}$.

Demonſtration.

$$\text{Let } y=\left\{\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & \\ & 2 & 3 & 4 & 5 & \\ 1 & & 3 & 4 & 5 & \text{continued to } n \text{ terms} \\ 1 & 2 & & 4 & 5 & \\ 1 & 2 & 3 & & 5 & \\ 1 & 2 & 3 & 4 & & \\ 1 & 2 & 3 & 4 & 5 & \end{array}\right\}=x.$$

It is evident by the ſcheme, that $y=1+3+6+10+15$, &c. *(n)* and alſo that x is $=1+4+9+16+25$, &c. *(n)* therefore $2y-x=1+2+3+4$, &c. $(n)=\overline{n+1}\times\frac{n}{2}$. Again, $x+y=\overline{n+1}\times\frac{n+1}{2}\times n$, whence $3y=\frac{n+1}{2}\times n+\overline{n+1}\times\frac{n+1}{2}\times n=\frac{n\times\overline{n+1}\times\overline{n+2}}{2}$ and $y=n\times\frac{n+1}{2}\times\frac{n+2}{3}$. *Q. E. D.*

Now to proceed to the ſolution of the problem. The chances for *(p)* points (when *p* points are required to be thrown it is to be underſtood of *p* points preciſely) with two ſuch dice are $p-1$, becauſe with $p-1$, $p-2$, $p-3$, *&c.* points upon either die, the ace, duce, tray, *&c.* of the other may be combined, for the ſame reaſon the chances for *(p)* points with three of thoſe dice are $1+2+3+4$, *&c.* $p-1$, equal to $\frac{p-1}{1}\times\frac{p-2}{2}$. Now as 3 is the leaſt number that can be thrown at one trial with three dice, it follows, that the chances for *(p)* points with four dice will be equal to $1+3+6+10+15$, *&c.* $p-3$, by writing 3, 4, 5, 6, *&c.* $p-1$ for *p* in the above expreſſion, whence by the lemma $\frac{p-1}{1}\times\frac{p-2}{2}\times\frac{p-3}{3}$, are the chances for *(p)* points with four dice, and thoſe for *(p)* points with five dice, will be found equal to $\frac{p-1}{1}\times\frac{p-2}{2}\times\frac{p-3}{3}\times\frac{p-4}{4}$, from hence the law of continuation is manifeſt.

Corollary.

The chances for *(p)* points with *(n)* ſuch dice is $\frac{p-1}{1}\times\frac{p-2}{2}\times\frac{p-3}{3}\times\frac{p-4}{4}$, *&c.* $(n-1)$

PROBLEM XLIV.

Let there be three dice; each die having (*p*) or a greater number of faces, imagine the points on each die where their number is greater than (*f*) to be red, the other points black, let one die be called A, another B, the third C; it is required to find the chances for bringing up (*p*) points in one throw with theſe dice, ſo that the black part of each die may be upwards.

SOLUTION.

The chances for (*p*) points ſo as to have the red part of the die A always upwards are equal to thoſe for $p-f$ points without reſtriction. For conceive the black part of the die A expunged, and the (*f*) following red faces each diminiſhed by (*f*), then it is evident all the chances that before ſuch diminution gave (*p*) points with the red part of A upwards, will now give ($p-f$), wherefore if we conceive the diminiſhed red part to become black, and the remaining red part ſufficiently augmented, the die A will be juſt as it was at firſt, conſequently the chances for (*p*) points with the red part of the die A upwards, are exactly equal to thoſe for ($p-f$)

points without reſtriction; proceeding in the ſame manner with B and C, it will appear that the number of chances for *(p)* points with the red parts of A and B upwards, is equal to the number of chances for $p - 2f$, without any reſtriction; and the chances for *(p)* points with the red parts of A, B, C, upwards, equal to thoſe for $p - 3f$ without reſtriction; the ſame is to be underſtood of 4, 5, &c. or any given number of dice.

The chances for *(p)* points, without any reſtriction upon red or black faces, are equal to $\frac{p-1}{1} \times \frac{p-2}{2}$ (by the *Corollary* to *Problem* XLIII.) from which ſubtracting the ſum of all the chances for thoſe points ſo as to have the red parts of all, or ſome of the dice upwards, there will remain thoſe for *(p)* points with black faces only, theſe things premiſed, let $p - f = a$, $p - 2f = b$, $p - 3f = c$, then $\frac{c-1}{1} \times \frac{c-2}{2}$, the number of chances for *(p)* points with the red parts of A, B and C upwards, ſubtracted from $\frac{b-1}{1} \times \frac{b-2}{2}$, thoſe for *(p)* points with the red parts of A, B, or more upwards, leaves $\frac{b-1}{1} \times \frac{b-2}{2} - \frac{c-1}{1} \times \frac{c-2}{2}$ the chances for *(p)* points with the red parts of A and B, and the black part of C upwards, but as the black part of C might have changed with either A or B, the laſt expreſſion $\frac{b-1}{1}\times\frac{b-2}{2}-\frac{c-1}{1}\times\frac{c-2}{2}$ being multiplied by 3 ſhews the number of chances for (*p*) points with one black and two red faces. From $\frac{a-1}{1}\times\frac{a-2}{2}$ the chances for (*p*) points with the red part of A or more upwards, ſubtract the ſum of the chances for the ſame points with all red faces, with A and B red, C black, with A and C red, B black, the remainder $\frac{a-1}{1}\times\frac{a-2}{2}-2\times\frac{b-2}{2}\times\frac{b-1}{1}+2\times\frac{c-1}{1}\times\frac{c-2}{2}-\frac{c-1}{1}\times\frac{c-2}{2}$ or $\frac{a-1}{1}\times\frac{a-2}{2}-2\times\frac{b-1}{1}\times\frac{b-2}{2}+\frac{c-1}{1}\times\frac{c-2}{2}$ being multiplied by the number of combinations of 1 in 3, *viz.* 3, gives the chances for (*p*) points with one red face, and two black ones, wherefore $3\times\frac{b-1}{1}\times\frac{b-2}{2}-3\times\frac{c-1}{1}\times\frac{c-2}{2}$, the chances for (*p*) points with one black face and not more, plus $3\times\frac{a-1}{1}\times\frac{a-2}{2}-3\times2\times\frac{b-1}{1}\times\frac{b-2}{2}$ thoſe for (*p*) points with two black faces preciſely, being ſubtracted from $\frac{p-1}{1}\times\frac{p-2}{2}$, the chances for (*p*) points without any reſtriction upon either red or black faces, leaves the chances for thoſe points with the black parts of A, B, and C, upwards, equal to

$$+\frac{p-1}{1}\times\frac{p-2}{2}\text{ into }1,$$
$$-\frac{a-1}{1}\times\frac{a-2}{2}\text{ into }3,$$
$$+\frac{b-1}{1}\times\frac{b-2}{2}\text{ into }3.$$
$$-\frac{c-1}{1}\times\frac{c-2}{2}\text{ into }1.$$

PROBLEM XLV.

Other things remaining as in the laſt problem, let it be required to find the chances for throwing (p) points in one throw with four dice.

SOLUTION.

Put $p-f=a$, $p-2f=b$, $p-3f=c$, $p-4f=d$, call the dice A, B, C, D. Now $\frac{p-1}{1}\times\frac{p-2}{2}\times\frac{p-3}{3}$ are the chances for (p) points with the propoſed dice without reſtriction (*Cor. Prob.* XLIII.) $\frac{c-1}{1}\times\frac{c-2}{2}\times\frac{c-3}{3}$ the chances for thoſe points, ſo as to have the red parts of A, B, C, or more upwards, $\frac{d-1}{1}\times\frac{d-2}{2}\times\frac{d-3}{3}$, thoſe for (p) points with all red faces upwards, hence we get $\frac{c-1}{1}\times\frac{c-2}{2}\times\frac{c-3}{3}-\frac{d-1}{1}\times\frac{d-2}{2}\times\frac{d-3}{3}$, the chances for (p) points with the black part of C, and red parts of A, B, and D, upwards, which being multiplied by 4, the given number of dice, gives the chances for (p) points with one black face, and the others red. Again, $\frac{b-1}{1}\times\frac{b-2}{2}\times\frac{b-3}{3}$ are the chances for *(p)* points with the red parts of A and B or more upwards, from which ſubtract the chances for *(p)* points with all red faces, and alſo the double of thoſe for the ſame points with A, B and C red, D black, (becauſe A, B and D, may come up red, and C black) there will remain the chances for *(p)* points, with the red parts of A and B, and the black parts of C and D upwards, equal to $\frac{b-1}{1}\times\frac{b-2}{2}\times\frac{b-3}{3}-2\times\frac{c-1}{1}\times\frac{c-2}{2}\times\frac{c-3}{3}+\frac{d-1}{1}\times\frac{d-2}{2}\times\frac{d-3}{3}$, but as there are the very ſame chances for any other two, than A and B, being red, it follows that the ſaid expreſſion being repeated as often as there are different combinations of 2 in 4, *viz.* 6, will give the number of chances for *p* points with two black faces, and two red ones upwards. Our next buſineſs will be to find the chances for *(p)* points ſo as to have one red face and not more upwards, which may be done thus, $\frac{d-1}{1}\times\frac{d-2}{2}\times\frac{d-3}{3}$ are the chances for *(p)* points with all red faces, $\frac{c-1}{1}\times\frac{c-2}{2}\times\frac{c-3}{3}-\frac{d-1}{1}\times\frac{d-2}{2}\times\frac{d-3}{3}$ are thoſe for *(p)* points with the red parts of A, B, C, and black part of D, which muſt be multiplied by 3, becauſe the red part of B or C, might have changed with D. Again, $\frac{b-1}{1}\times\frac{b-2}{2}\times\frac{b-3}{3}-2\times\frac{c-1}{1}\times\frac{c-2}{2}\times\frac{c-3}{3}+\frac{d-1}{1}\times\frac{d-2}{2}\times\frac{d-3}{3}$ are the chances for (*p*) points with A and B red, C and D black, which muſt be repeated three times, becauſe the red part of B might have changed with C or D, now it is evident that if the ſum of theſe three expreſſions be ſubtracted from the number of chances for (*p*) points with the red parts of A upwards, without any reſtriction upon the others, B, C, or D, *viz.* from $\frac{a-1}{1}\times\frac{a-2}{2}\times\frac{a-3}{3}$, the remainder $\frac{a-1}{1}\times\frac{a-2}{2}\times\frac{a-3}{3}-\frac{d-1}{1}\times\frac{d-2}{2}\times\frac{d-3}{3}+3\times\frac{c-1}{1}\times\frac{c-2}{2}\times\frac{c-3}{3}-3\times\frac{b-1}{1}\times\frac{b-2}{2}\times\frac{b-3}{3}$ will be the number of chances for (*p*) points ſo as to have the red part of the die A up, without any other of the ſame colour, which multiplied by 4 gives the chances for (*p*) points with one red face and three black ones. By collecting together theſe ſeveral chances we get,

1 into $\frac{d-1}{1}\times\frac{d-2}{2}\times\frac{d-3}{3}$ for (*p*) points with all red faces.

4 into $\frac{c-1}{1}\times\frac{c-2}{2}\times\frac{c-3}{3}-\frac{d-1}{1}\times\frac{d-2}{2}\times\frac{d-3}{3}$ } for (*p*) points with three red and one black.

6 into

$$\left.\begin{aligned}&6 \text{ into } \tfrac{b-1}{1}\times\tfrac{b-2}{2}\times\tfrac{b-3}{3}+\\&\tfrac{d-1}{1}\times\tfrac{d-2}{2}\times\tfrac{d-3}{3}-2\times\\&\tfrac{c-1}{1}\times\tfrac{c-2}{2}\times\tfrac{c-3}{3}\end{aligned}\right\}\text{with two red and two black.}$$

$$\left.\begin{aligned}&4 \text{ into } 3\times\tfrac{c-1}{1}\times\tfrac{c-2}{2}\times\tfrac{c-3}{3}+\\&\tfrac{a-1}{1}\times\tfrac{a-2}{2}\times\tfrac{a-3}{3}-3\times\tfrac{b-1}{1}\\&\times\tfrac{b-2}{2}\times\tfrac{b-3}{3}-\tfrac{d-1}{1}\times\tfrac{d-2}{2}\times\\&\tfrac{d-3}{3}.\end{aligned}\right\}\text{with one red and three black.}$$

The ſum of theſe being ſubtracted from $\frac{p-1}{1}\times\frac{p-2}{2}\times\frac{p-3}{3}$, the number of chances for (p) points with four dice without any reſtriction, will leave

$$\begin{aligned}&+\tfrac{p-1}{1}\times\tfrac{p-2}{2}\times\tfrac{p-3}{3}\times 1\\&-\tfrac{a-1}{1}\times\tfrac{a-2}{2}\times\tfrac{a-3}{3}\times 4\\&+\tfrac{b-1}{1}\times\tfrac{b-2}{2}\times\tfrac{b-3}{3}\times 6\\&-\tfrac{c-1}{1}\times\tfrac{c-2}{2}\times\tfrac{c-3}{3}\times 4\\&+\tfrac{d-1}{1}\times\tfrac{d-2}{2}\times\tfrac{d-3}{3}\times 1\end{aligned}$$

The number of chances for throwing (p) points preciſely, in one throw with four dice, ſo as the black part of each may be upwards. Now let the given number of dice be called n,

n, then from this, and the preceding problem, the law of continuation may be eaſily deduced; for the number of factors in each line is $n-1$, and their correſponding numerical multipliers form the unciæ of a binomial $1+1$, raiſed to the n power, whence this general theorem,

$$+\frac{p-1}{1}\times\frac{p-2}{2}\times\frac{p-3}{3}\,(n-1)$$

$$-\frac{a-1}{1}\times\frac{a-2}{2}\times\frac{a-3}{3}\,(n-1)\text{ into }n$$

$$+\frac{b-1}{1}\times\frac{b-2}{2}\times\frac{b-3}{3}\,(n-1)\text{ into }n\times\frac{n-1}{2}$$

$$-\frac{c-1}{1}\times\frac{c-2}{2}\times\frac{c-3}{3}\,(n-1)\text{ into }n\times\frac{n-1}{2}\times\frac{n-2}{3}$$

$$+\frac{d-1}{1}\times\frac{d-2}{2}\times\frac{d-3}{3}\,(n-1)\text{ into }n\times\frac{n-1}{2}\times\frac{n-2}{3}$$

$$\times\frac{n-3}{4}$$

&c. *&c.*

Wherein $a=p-f$, $b=p-2f$, $c=p-3f$, $d=p-4f$, *&c.*

Note, The terms of the ſeries are to be continued till ſome one factor becomes nothing or negative.

⁂

Otherwiſe.

Suppoſe the faces of each die to be marked r^1, r^2+r^3, r^4, *&c.* progreſſively, to as many terms as are the number of faces upon each, then to have all the chances upon two ſuch dice, $r^1+r^2+r^3+r^4$, *&c.* *(f)* (*f* being the number of faces upon each die) muſt be ſquared, and for all the chances upon three, the ſaid expreſſion muſt be cubed, whence to get all the chances upon any aſſigned number *(n)* of thoſe dice, we muſt raiſe $r^1+r^2+r^3+r^4$, *&c.* . . . *(f)* to the *n* power. Now in order to obtain the chances of any intermediate number of points, between the leaſt *(n)* and the greateſt f^n, that can be thrown with any given number of dice (ſuppoſe three) we muſt raiſe $r^1+r^2+r^3+r^4$, *&c.* . . . r^8 (*f* being ſuppoſed $= 8$) to the third power, which being done, and as many terms as are neceſſary for our purpoſe ſelected therefrom, will ſtand thus,

$$\begin{array}{rrrrrrrrrrrrrl}
r^3+ & 2r^4+ & 3r^5+ & 4r^6+ & 5r^7+ & 6r^8+ & 7r^9+ & 8r^{10}+ & 7r^{11}+ & 6r^{12}+ & 5r^{13}+ & 4r^{14}+ & 3r^{15} & \\
 & r^4+ & 2r^5+ & 3r^6+ & 4r^7+ & 5r^8+ & 6r^9+ & 7r^{10}+ & 8r^{11}+ & 7r^{12}+ & 6r^{13}+ & 5r^{14}+ & 4r^{15} & \\
 & & r^5+ & 2r^6+ & 3r^7+ & 4r^8+ & 5r^9+ & 6r^{10}+ & 7r^{11}+ & 8r^{12}+ & 7r^{13}+ & 6r^{14}+ & 5r^{15} & \\
 & & & r^6+ & 2r^7+ & 3r^8+ & 4r^9+ & 5r^{10}+ & 6r^{11}+ & 7r^{12}+ & 8r^{13}+ & 7r^{14}+ & 6r^{15} & \&c. \\
 & & & & r^7+ & 2r^8+ & 3r^9+ & 4r^{10}+ & 5r^{11}+ & 6r^{12}+ & 7r^{13}+ & 8r^{14}+ & 7r^{15} & \\
 & & & & & r^8+ & 2r^9+ & 3r^{10}+ & 4r^{11}+ & 5r^{12}+ & 6r^{13}+ & 7r^{14}+ & 8r^{15} & \\
 & & & & & & r^9+ & 2r^{10}+ & 3r^{11}+ & 4r^{12}+ & 5r^{13}+ & 6r^{14}+ & 7r^{15} & \&c. \\
 & & & & & & & r^{10}+ & 2r^{11}+ & 3r^{12}+ & 4r^{13}+ & 5r^{14}+ & 6r^{15} & \\
\hline
r^3+ & 3r^4+ & 6r^5+ & 10r^6+ & 15r^7+ & 21r^8+ & 28r^9+ & 36r^{10}+ & 42r^{11}+ & 46r^{12}+ & 48r^{13}+ & 48r^{14}+ & 46r^{15} &
\end{array}$$

In this ſeries the ſum of the co-efficients of the ſame powers of r, leſs than 10, is equal to $1+2+3+4$, &c. continued to as many terms as are denoted by the given power leſs 2, whence if p be put for any ſuch power of r, the co-efficient of r^p will be $\frac{p-1}{1}\times\frac{p-2}{2}$. Now to find the co-efficient of any other term when (p) the exponent of r is greater than 10, as for example, the term r^{15}. The ſum of the ſeveral co-efficients of r^{15} is $6+7+8+7+6+5+4+3$, that is equal to $1+2+3+4+5+6+7+8+9+10+11+12+13-\overline{1+2+3+4+5}-\overline{2+4+6+8+10}$, but $1+2+3+4$, &c. . . . 13, is equal to $\frac{p-1}{1}\times\frac{p-2}{2}$, and $1+2+3+4+5+2+4+6+8+10$ or $\overline{1+2+3+4+5}\times 3$ is equal to $\overline{p-f-2+1}\times\frac{p-f-2}{2}$ or (putting $p-f=a$) $\frac{a-1}{1}\times\frac{a-2}{2}$ multiplied by 3, which ſubtracted from $\frac{p-1}{1}\times\frac{p-2}{2}$ leaves $\frac{p-1}{1}\times\frac{p-2}{2}-\overline{\frac{a-1}{1}\times\frac{a-2}{2}}\times 3$, the required co-efficient of r^p, $=46$. If the co-efficient of any other term greater than 16 or $2f$ were required, it will, by proceeding in the ſame manner, be found equal to (putting $p-2f=b$)

$\frac{p-1}{1}$

$$\frac{p-1}{1}\times\frac{p-2}{2}$$
$$-\frac{a-1}{1}\times\frac{a-2}{2}\times 3$$
$$+\frac{b-1}{1}\times\frac{b-2}{2}\times 3$$

If the ſeries $r^1+r^2+r^3+r^4$, *&c.* . . *(f)* be raiſed to the fourth power, and the neceſſary terms ſelected thereform as before, the coefficient of any term of that power as r^p, or the chances for throwing *(p)* points with four dice will (putting $p-3f=c$, $p-4f=d$) be found equal to

$$\frac{p-1}{1}\times\frac{p-2}{2}\times\frac{p-3}{3}\times 1$$
$$-\frac{a-1}{1}\times\frac{a-2}{2}\times\frac{a-3}{3}\times 4$$
$$+\frac{b-1}{1}\times\frac{b-2}{2}\times\frac{b-3}{3}\times 6$$
$$-\frac{c-1}{1}\times\frac{c-2}{2}\times\frac{c-3}{3}\times 4$$
$$+\frac{d-1}{1}\times\frac{d-2}{2}\times\frac{d-3}{3}\times 1$$

Which being the very ſame expreſſion as was determined by the former method, it follows that the ſame general rule would reſult from this inveſtigation as before, and therefore need not be here repeated.

Example I.

Let it be required to find how many chances there are for throwing 16 points preciſely, with four common dice. Here $p=16$, $n=4$, $f=6$, $p-f=10$, $p-2f=4$, whence $\frac{15}{1}\times\frac{14}{2}\times\frac{13}{3}-\frac{9}{1}\times\frac{8}{2}\times\frac{7}{3}\times4+\frac{3}{1}\times\frac{2}{2}\times\frac{1}{3}\times6=125$, the chances required.

Example II.

Let it be required to find the chances for throwing 27 points preciſely, with ſix dice. Here $p=27$, $n=6$, $f=6$, $a=21$, $b=15$, $c=9$, $d=3$,

$$\text{whence}\begin{cases}+\frac{26}{1}\times\frac{25}{2}\times\frac{24}{3}\times\frac{23}{4}\times\frac{22}{5}=+65780\\-\frac{20}{1}\times\frac{19}{2}\times\frac{18}{3}\times\frac{17}{4}\times\frac{16}{5}\times\frac{6}{1}=-93024\\+\frac{14}{1}\times\frac{13}{2}\times\frac{12}{3}\times\frac{11}{4}\times\frac{10}{5}\times\frac{6}{1}\times\frac{5}{2}=+30030\\-\frac{8}{1}\times\frac{7}{2}\times\frac{6}{3}\times\frac{5}{4}\times\frac{4}{5}\times\frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}=-1120\end{cases}$$

and 1666 the number of chances ſought.

Note, As all the points equally diſtant from the extremes, that is from the leaſt and greateſt number of points that can be thrown with the propoſed dice, have exactly the ſame chances for happening, it will be convenient if the given number of points be nearer the greater extreme than to the leſs, to uſe inſtead thereof that number which is as far diſtant from the leſſer extreme, as the given one is from the greater.

Example III.

How many chances are there to have 50 points preciſely, on nine dice. Here 13 being

ing as much above 9 the leſſer extreme, as 50 is exceeded by 54 the greater extreme, we have $p = 13$, $n = 9$, $f = 6$, $a = 7$, $b = 1$, therefore $\frac{13}{1}\times\frac{12}{2}\times\frac{11}{3}\times\frac{10}{4}\times\frac{9}{5}\times\frac{10}{6}\times\frac{7}{7}\times\frac{6}{8} - \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}\times\frac{2}{5}\times\frac{1}{6}\times\frac{9}{1}$, or 1278 is the number required. If this number be divided by 6 raiſed to the ninth power, *viz.* 10077696, the quotient $\frac{1278}{10077696}$ ſhews the probability of throwing in one caſt with nine dice 50 points preciſely, whence the odds as 7885 to 1 very near.

The chances for (p) or a greater number of points in one throw with (n) dice, may be found by collecting together the reſpective chances for throwing preciſely $p-1$ and all its inferior points down to (n), and ſubtracting their ſum from f^n. Now the ſum of all the chances for $p-1$, $p-2$, $p-3$, &c. down to (n) incluſive, is equal to

$\frac{p-1}{2}\times\frac{p-2}{3}\,(n)$

$-\frac{a-1}{2}\times\frac{a-2}{3}\,(n)$ into n

$+\frac{b-1}{2}\times\frac{b-2}{3}\,(n)$ into $n\times\frac{n-1}{2}$

$-\frac{c-1}{2}\times\frac{c-2}{3}\,(n)$ into $n\times\frac{n-1}{2}\times\frac{n-2}{3}$

$+\frac{d-1}{2}\times\frac{d-2}{3}\,(n)$ into $n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$

&c. &c.

Which ſubtracted from f^n leaves the number of chances for (p) or a greater number of points.

Example IV.

What are the odds that in one throw with ſix dice there ſhall come up 15 points, or a greater number? Here $n=6$, $p=15$, $a=9$, $b=3$, theſe being ſubſtituted in the general expreſſion, it becomes equal to 405, the number of chances by which 14 points and all its inferior numbers may happen, this taken from $6^6=46656$ leaves 46251, whence the odds as $114\frac{1}{5}$ to 1.

PROBLEM XLVI.

To find the probability of throwing preciſely two faces of one ſort, and three faces of another ſort, in one throw with five common dice.

SOLUTION.

Imagine the dice to be thrown ſingly, then as the required probability cannot be any ways affected by whatever the firſt throw may happen to be, $\frac{6\times5}{6^5}$ will expreſs the probability of throwing three faces of one ſort, and two of another ſort, in five throws with a ſingle die, but under ſuch reſtriction, that the faces of that ſort which may happen to be thrown firſt, do come up ſucceſſively, therefore $\frac{6 \times 5}{6^5}$ being divided by 1×2, *viz.* the changes that can be made with two quantities, gives $\frac{15}{6^5}$ for the probability of throwing the required faces, but according to ſome aſſigned order, ſuppoſe that in which they are propoſed in the problem. Now by *Problem* III. it appears that the probability of taking 5 things, conſiſting of 3 of one ſort, and 2 of another, in the ſame order in which they are wrote, is $\frac{3.2.1}{5.4.3}$, therefore $\frac{5.4.3}{3.2.1}$ is the number of different orders in which they can be taken, and conſequently $\frac{15}{6^5} \times \frac{5.4.3}{3.2.1}$ is the probability of throwing three faces of one ſort, and two of another, in five throws, but upon this ſuppoſition, that the faces of each ſort are conſtantly thrown with the ſame dice; now as the number of faces of each ſort that are to be thrown are unequal, thoſe dice with which the two faces of one ſort are ſuppoſed to be thrown, might have changed with two of thoſe with which the three faces may be thrown of the other ſort, ſo that the above found probability $\frac{15}{6^5} \times \frac{5.4.3}{1.2.3}$ is but $\frac{1}{2}$ the required, therefore being multiplied by 2 gives the probability ſought equal to $\frac{300}{7776}$.

PROBLEM XLVII.

Let there be a ſolid die having (m) ſimilar and equal faces, and thoſe faces numbered progreſſively 1, 2, 3, 4, &c. it is required to find the probability of throwing preciſely (p) faces of one ſort, (q) of another, (r) of a third, (s) of a fourth, &c. in one throw with a given number (S) of ſuch dice.

SOLUTION.

Put R for the number of ſorts of faces, P for as many of thoſe ſorts as are denoted by unequal quantities. Now by ſuppoſing the dice thrown ſingly, we have $\frac{m}{m^p} \times \frac{m-1}{m^q} \times \frac{m-2}{m^r}$, &c. (R) for the probability of throwing preciſely (p) faces of one ſort, (q) of another ſort, (r) of a third, (s) of a fourth, &c. but in ſuch manner, that the aſſigned number of faces of each ſort may come up ſucceſſively, without the interpoſition of any other ſort until that number be compleated; and as this probability is not any ways influenced by the order in which they may happen to ariſe, it follows that the quotient of $\frac{m}{m^p}\times\frac{m-1}{m^q}\times\frac{m-2}{m^r}$, &c. (R) divided by 1, 2, 3, 4, &c. (R) will be the probability of throwing the aſſigned number of each ſort of faces in ſome determinate order, as ſuppoſe that in which they are propoſed, *viz.* *(p)* firſt, *(q)* next, &c. and by *Problem* III. we have ——

$$\frac{1,2,3,\&c.\ (q)\text{ into }1,2,3,\&c.\ (r)\text{ into }1,2,3,\&c.\ (s)\ \&c.}{S\times\overline{S-1}\times\overline{S-2}\times\overline{S-3}\times\overline{S-4},\ \&c.\ (S-p)}$$

for the probability of their being thrown in that order, conſequently ———

$$\frac{S\times\overline{S-1}\times\overline{S-2}\times\overline{S-3}\times\overline{S-4},\ \&c.\ (S-p)}{1,2,3,\&c.\ (q)\text{ into }1,2,3,\&c.\ (r)\text{ into }1,2,3,\&c.\ (s)\ \&c.}$$

expreſſes the number of variations that can be made with the quantities $p, q, r, s,$ &c. whence

$$\frac{\frac{m}{m^p}\times\frac{m-1}{m^q}\times\frac{m-2}{m^r},\ \&c.\ (R)}{1,2,3,4,5,\&c.\ (R)}$$ multiplied by ——

$$\frac{S\times\overline{S-1}\times\overline{S-2}\times\overline{S-3},\ \&c.\ (S-p)}{1,2,3,\&c.\ (q)\text{ into }1,2,3,\&c.\ (r)\text{ into }1,2,3.\&c.\ (s)}$$

is the probability of ſucceeding, under this reſtriction, *viz.* that the faces of each are conſtantly thrown with the ſame dice, but as the ſeveral parcels of dice $p, q, r, s,$ &c. (where the number of each is unequal) may mutually change among themſelves, it follows that the laſt mentioned probability muſt be repeated as often as there are changes to be made by the unequal quantities $p, q, r, s,$ &

&c. that $R \times \overline{R-1} \times \overline{R-2} \times \overline{R-3}$, *&c.* (P) to give the probability required equal to

$$\frac{\frac{m}{m^q} \times \frac{m-1}{m^q} \times \frac{m-2}{m^r}, \,\&c.\ (R)}{1, 2, 3, 4, 5, \&c. \ldots (R)} \times \frac{S \times \overline{S-1} \times \overline{S-2} \times \overline{S-3}, \,\&c.\ (S-p)}{1, 2, 3, \&c.\ (q) \times 1, 2, 3, \&c.\ (r) \times 1, 2, 3, \&c.\ (s)\ \&c.} \times R \times \overline{R-1} \times \overline{R-2}, \,\&c.\ (P).$$

Example I.

Required the probability of throwing precisely two faces of one ſort, three faces of another ſort, in one throw with ſeven common dice. Here $m=6$, $S=7$, $p=2$, $q=3$, $r=1$, $s=1$, $R=4$, and $P=2$, whence the required probability is $\frac{6}{6^2} \times \frac{5}{6^3} \times \frac{4}{6} \times \frac{3}{6} \times \frac{7.6.5.4.3}{1.2.1.1.3} \times \frac{4.3}{1.2.3.4} = \frac{75600}{279936}$ or $\frac{35}{1296}$.

Example II.

What is the probability of throwing precisely four faces of ſome one ſort, in one throw with ſix common dice. Here $m=6$, $S=6$, $p=4$, $q=1$, $r=1$, $R=3$, $P=1$, the probability required $= \frac{1800}{46656}$ or $\frac{50}{1296}$.

Example III.

It is required to find the probability of throwing three or a greater number of faces, of ſome one ſort, in five throws with a ſingle die. The probability of bringing up a different face each throw is $\frac{5}{54}$, which ſubtracted from unity leaves $\frac{49}{54}$, that of bringing up two faces or more of ſome one ſort, and again ſubtracting the probability for throwing preciſely two faces of one ſort, *viz.* $\frac{25}{54}$, the remainder $\frac{24}{54}$ or $\frac{4}{9}$, is the probability required.

PROBLEM XLVIII.

Any number *(n)* of letters, *a*, *b*, *c*, *d*, *e*, *f*, *g*, *&c.* or things repreſented by them, being placed according to the order or rank they obtain in the alphabet, and afterwards taken one by one as it happens; to find the probability that of thoſe letters ſome aſſigned ones ſhall be found in their proper places, and that others of them ſhall be diſplaced.

SOLUTION.

Suppoſe the given letters to be *a*, *b*, *c*, *d*, *e*, *f*, and that it were required to find the probability that *a*, *b* and *c*, ſhall be drawn out of their proper places, without any regard to the remaining letters being in or out of theirs. The probability that *b* ſhall be taken out of its proper place is $\frac{5}{6}$, that of *a* being in its proper place and *b* out of his is $\frac{1}{6} \times \frac{4}{5}$, which ſubtracted from $\frac{5}{6}$, the probability that *b* ſhall be found diſplaced, leaves $\frac{7}{10}$, the probability that both *a* and *b* ſhall be diſplaced, this is evident, for the probability that *b* ſhall be taken out of its proper place without reſtriction, certainly contains the probability whereby *a* and *b* may be both diſplaced, together with that of *a* being in its proper place and *b* diſplaced. Now the probability that *a* and *b* may be both diſplaced being the very ſame as that of any other two letters, it follows that $\frac{7}{10}$ is alſo the probability of *b* and *c* being drawn out of their proper places, from which ſubtract the probability of *a* being taken in its proper place, and *b* and *c* out of theirs, *viz.* $\frac{13}{180}$, the remainder $\frac{71}{120}$ is the probability of *a*, *b*, and *c*, being all three diſplaced. By the ſame way of proceeding we may find the probability that any other letters, as *a*, *b*, *c*, *d*, may be all taken out of their proper places, by ſubtracting from the probability laſt found, that of *a* being found in its proper place, and at the ſame time *b*, *c* and *d* out of theirs. But in order to find the law of continuation by which the inveſtigation may become general, let (the other things remaining as before) *p* be the number of aſſigned letters to be diſplaced. The probability that *b* ſhall be diſplaced without any reſtriction upon the remaining letters, is $1 - \frac{1}{n}$, that of *a* being drawn in its proper place, and *b* out of his, is $\frac{1}{n}\times\overline{1-\frac{1}{n-1}}$, this taken from $1-\frac{1}{n}$, leaves $1-\frac{2}{n}+\frac{1}{n\times\overline{n-1}}$, the probability that *a* and *b*, or any other two letters, ſhall be diſplaced. If *a* ſhould be drawn firſt, of which the probability is $\frac{1}{n}$, that of *b* and *c* being afterwards diſplaced, will be $1-\frac{2}{n-1}+\frac{1}{\overline{n-1}\times\overline{n-2}}$, which multiplied by $\frac{1}{n}$, and the product ſubtracted from $1-\frac{2}{n}+\frac{1}{n\times\overline{n-1}}$, *viz.* the probability that *b* and *c* ſhall be both diſplaced without any regard to *a* being in or out of its proper place, leaves $1-\frac{3}{n}+\frac{3}{n\times\overline{n-1}}-\frac{1}{n\times\overline{n-1}\times\overline{n-2}}$, the required probability, or that of *a*, *b*, *c*, or any other three letters being drawn out of their proper places. If from this laſt found probability, which is alſo that of *b*, *c*, *d*, happening out of their proper places, we ſubtract the probability that *a* ſhall happen in its proper place, and *b*, *c*, *d*, at the ſame time out of theirs, that is, $\frac{1}{n}-\frac{3}{n\times\overline{n-1}}+\frac{3}{n\times\overline{n-1}\times\overline{n-2}}-\frac{1}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}$ the remainder $1-\frac{4}{n}+\frac{6}{n\times\overline{n-1}}-\frac{4}{n\times\overline{n-1}\times\overline{n-2}}+\frac{1}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}$ gives the probability that *a*, *b*, *c*, *d*, or any other four letters ſhall all happen out of their proper places. From hence the law of continuation is manifeſt, the numerators being the unciæ of the (p) power of a binomial, and the denominators, 1, n, $n\times\overline{n-1}$, $n\times\overline{n-1}\times\overline{n-2}$, $n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}$, and therefore the probability that (p) aſſigned letters ſhall all be taken out of their proper places is $1-\frac{p}{n}+\frac{p\times\frac{\overline{p-1}}{2}}{n\times\overline{n-1}}-\frac{p\times\frac{\overline{p-1}}{2}\times\frac{\overline{p-2}}{3}}{n\times\overline{n-1}\times\overline{n-2}}+\frac{p\times\frac{\overline{p-1}}{2}\times\frac{\overline{p-2}}{3}\times\frac{\overline{p-3}}{4}}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}$, &c. $(p+1)$. The probability that any aſſigned letters a, b, c, d, &c. whoſe number is (q) ſhall fall in their proper places is $\frac{1}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}}\ (q)$ (*Problem* I.) wherefore the probability that (p) aſſigned letters ſhall be taken out, and (q) other aſſigned ones in their proper places, will (by writing $n-q$ for n in the former probability) be expreſſed by $\frac{1}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\ (q)}$ into $1-\frac{p}{n-q}+\frac{p\times\frac{\overline{p-1}}{2}}{\overline{n-q}\times\overline{n-q-1}}-\frac{p\times\frac{\overline{p-1}}{2}\times\frac{\overline{p-2}}{3}}{\overline{n-q}\times\overline{n-q-1}\times\overline{n-q-2}}\ (p+1)$.

Corollary I.

If $p+q=n$, the laſt mentioned probability becomes $\frac{1}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\ (q)}$ into $1-1+\frac{1}{2}-\frac{1}{2,3}+\frac{1}{2,3,4}\ (n-q+1)$ equal to

$\frac{1}{n\times\overline{n-1}+\overline{n-2}\times\overline{n-3}\,(q)}$ into $\frac{1854}{5040}$ or $\frac{,367857}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\,(q)}$ by taking only ſix terms of the fractions.

Corollary II.

If a given number *(q)* of letters be to be taken in their proper places, and the reſt out of theirs, then the probability of taking *(q)* aſſigned ones in their proper places, and the reſt otherwiſe, being $\frac{1}{n\times\overline{n-1}\times\overline{n-2}\,(q)}$ into $1-1+\frac{1}{2}-\frac{1}{2,3}+\frac{1}{2,3,4}\,(n-q+1)$ it follows, that as often as *(q)* things can be taken or combined in *(n)*, that is $n\times\frac{n-1}{2}\times\frac{n-2}{3}\,(q)$ juſt ſo often muſt the aforeſaid probability be repeated to give the true value, hence $n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}\times\frac{\overline{n-3}}{4}\,(q)\times\frac{,367857}{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\,(q)}$ or its equal $\frac{,367857}{1.2.3.4.5(q)}$ is the probability required, very nearly.

PROBLEM XLIX.

To find the probability of an event happening a given number of times *(p)* without intermiſſion, in a given number *(n)* of trials, not exceeding $2p$.

SOLUTION.

Suppoſe it were required to find the probability that the propoſed event ſhall happen three times ſucceſſively in ſix trials, the chances for its happening and failing in each trial being (a) and (b) reſpectively. It is evident that if the event happens three times ſucceſſively in ſix trials it muſt be either the 1st, 2d, 3d, 2d, 3d, 4th, 3d, 4th, 5th, or 4th, 5th, 6th, and alſo that the trial preceeding next to the 2d, the 3d, or 4th, muſt fail. Now let the given number of trials be wrote down as in the table here annexed, wherein the third column contains thoſe trials in which the event muſt neceſſarily happen, the ſecond thoſe in which it muſt neceſſarily fail, and the firſt column contains thoſe trials in which it is indifferent whether the event happens or fails, the quantities in the fourth column are the correſponding probabilities to the trials in the other columns, as the probability of happening three times ſucceſſively is $\frac{a^3}{\overline{a+b}^3}$, or $\frac{a^3}{d^3}$ (making $a+b=d$) that of failing the firſt trial, and happening the ſecond, third, and fourth, $\frac{b}{d}\times\frac{a^3}{d^3}$ and ſo of the others. The ſum of theſe, *viz.* $1+\frac{b}{d}+\frac{ab+bb}{d^2}+\frac{aab+2abb+b^3}{d^3}$ into $\frac{a^3}{d^3}$ or $\overline{\overline{1+1+1}\times\frac{b}{d}+1}\times\frac{a^3}{d^3}$ is the probability required, equal to $\overline{\overline{n-p}\times\frac{b}{d}+1}\times$

$\frac{a^p}{d^p}$, n being the given number of trials provided it be not greater than $2p$.

Column I. Trials.	Column II. Trials.	Column III. Trials.	Probabilities.
0	0	1st, 2d, 3d	$\frac{a^3}{d^3}$
0	1st	2d, 3d, 4th	$\frac{a^3}{d^3} \times \frac{b}{d}$
1st	2d	3d, 4th, 5th	$\frac{a^3}{d^3} \times \frac{a}{d} \times \frac{b}{d} + \frac{a^3}{d^3} \times \frac{b}{d} \times \frac{b}{d}$
1st, 2d	3d	4th, 5th, 6th	$\frac{a^3}{d^3} \times \frac{a^2b}{d^3} + \frac{2a^3}{d^3} \times \frac{ab^2}{d^3} + \frac{b^3}{d^3}$

Example I.

Let it be required to find the probability of throwing a propoſed chance 5 times without intermiſſion in 9 trials, when the odds for its happening at any aſſigned trial is as 2 to 1, then $a=2$, $b=1$, $p=5$, $n=9$, and $\overline{n-p\times\frac{b}{d}+1}\times\frac{a^p}{d^p}=\frac{224}{729}$.

Example

Example II.

What is the probability of throwing with a ſingle die the ace 7 times ſucceſſively in 11 trials. Here $a=1$, $b=5$, $p=7$, $n=11$, whence $\frac{13}{839808}$ is the probability required.

PROBLEM L.

To find the probability that a propoſed event ſhall happen *(p)* times ſucceſſively, once or oftener, in a given number *(n)* of trials.

SOLUTION.

Let it be required to find the probability of winning 3 times ſucceſſively in 7 trials, ſuppoſe the probability of winning 3 times ſucceſſively to be denoted by e, that of failing any aſſigned trial by f, put $fe=y$, then the probability of ſucceeding in ſix trials is $e+3y$, by the laſt problem, but the propoſed event (*viz.* that of winning 3 times ſucceſſively) may happen twice in 7 trials, for it may happen in the firſt three, and again in the laſt four. The probability that the event may take place in the firſt three trials, is included in $e+3y$, therefore $\overline{1-e}\times y$, the probability that the event fails during the firſt three trials, and happens in the laſt four, being added to the former, gives $e+4y-ey$, the probability of winning three times ſucceſſively, once or oftener, in 7 trials. Let the number of trials be 8. Now if the event happens twice in 8 trials, it muſt be during the firſt 4 trials, and alſo in the laſt 4, but the probability of its happening during the firſt 4, is included in $e+4y-ey$, therefore $\overline{1-e-y}\times y$, which is the probability of its failing in the firſt 4 trials, and happening in the laſt 4, being added to $e+4y-ey$ gives $e+5y-2ey-yy$ the probability required, and the ſame method is to be obſerved for any other given number of trials. But to render this inveſtigation more general, let us take $n=2p+1$, that is, for 3 write p, and for 7 write $p+1$, then the probability of the event happening once or oftener in 7 trials, will be expreſſed by $e+\overline{p+1}\times y-ey$, in 8 trials, or $2p+2$, by $e+\overline{p+2}\times y-2ey-yy$, as determined above. Let $n=2p+3$, if the event happens twice in theſe trials it muſt be during the firſt $p+2$, and laſt $p+1$ trials, but the probability of its happening during the firſt $p+2$ trials, is included in that of its happening in $2p+2$ trials, wherefore $\overline{1-e-2y}\times y$, the probability of its failing in the firſt $p+2$ trials, and happening in the laſt $p+1$, being added to $e+\overline{p+2}\times y-2ey$ $-yy$

$-yy$ gives $e+\overline{p+3}\times y-3ey-3yy$, the probability that the event happens once or oftener in $2p+3$ trials. It may be proper here to obſerve, that the quantities with the affirmative ſigns in each of the foregoing probabilities, as $e+\overline{p+1}\times y$ in the firſt, $e+\overline{p+2}\times y$ in the ſecond, and $e+\overline{p+3}\times y$ in the third, are each equal to $\overline{\overline{n-p}\times\frac{b}{d}+1}\times\frac{a^p}{d^p}$ or $\overline{n-p}\times y+e$ (determined by the laſt problem) without any reſtriction upon the value of (n), therefore ey, $2ey+yy$, $3ey+3yy$, are the quantities that muſt be reſpectively ſubtracted from thoſe to give the probability required. Upon theſe principles the following table was conſtructed, wherein the firſt column towards the left hand ſhews the quantities that are to be ſubtracted, and the ſecond column the given number of trials. Now by inſpecting this table the law of continuation will become manifeſt, for (n) being the given number of trials, $\overline{n-2p}\times ey$ will expreſs the ſum of all the ey's as far as are neceſſary, thoſe quantities not taking place until after $2p$ trials, for which reaſon cyphers are placed in the firſt column againſt the trials in the ſecond column, leſs than $2p+1$. The ſum of all the yy's is $\overline{n-2p-1}\times\frac{n-2p}{2}\times yy$, for they form an arithmetical progreſſion, whoſe firſt term is 0, common difference 1, and number of terms $n-2p$. The ſum of the yyy's is $n-3p \times \frac{n-3p-1}{2} \times \frac{n-3p-2}{3} \times y^3$, and in like manner may the reſpective ſums of the other quantities as ey^3, y^4, ey^4, &c. be obtained, therefore if A, B, C, D, &c. be put for $n-2p$, $n-3p$, $n-4p$; $n-5p$, &c. we ſhall ſhall have the following ſeries, *viz.* $A \times ey + A \times \frac{A-1}{2} \times yy - B \times \frac{B-1}{2} \times eyy - C \times \frac{C-1}{2} \times \frac{C-2}{3} \times y^3 + D \times \frac{D-1}{2} \times \frac{D-2}{3} \times ey^3 + E \times \frac{E-1}{2} \times \frac{E-2}{3} \times \frac{E-3}{4} \times y^4$, &c. which being ſubtracted from $\overline{\overline{}}_{n-p} \times \overline{\frac{b}{d}+1} \times \frac{a^p}{d^p}$, or its equal $\overline{}_{n-p} \times y + e$ gives the probability that the propoſed event may happen once, at leaſt, in (n) trials.

p	0
$p+1$	0
$p+2$	0
$p+3$	0
$2p$	a
$2p+1$	ey
$2p+2$	$ey+yy$
$2p+3$	$ey+2yy$
$3p$	$ey+\overline{p-1}\times yy$
$3p+1$	$ey+p\times yy$
$3p+2$	$ey+\overline{p+1}\times yy-eyy$
$3p+3$	$ey+\overline{p+2}\times yy-2eyy-y^3$
$3p+4$	$ey+\overline{p+3}\times yy-3eyy-3y^3$
$4p$	$ey+\overline{2p-1}\times yy-\overline{p-1}\,eyy-p-1\times$ $\frac{p-2}{2}y^3$
$4p+1$	$ey+2p\times yy-peyy-p\times\frac{p-1}{2}y^3$
$4p+2$	$ey+\overline{2p+1}\times yy-\overline{p+1}\,eyy-\overline{p+1}\times\frac{p}{2}y^3$
$4p+3$	$ey+\overline{2p+2}\times yy-\overline{p+2}\,eyy-\overline{p+2}\times$ $\frac{p+1}{2}y^3+ey^3$
$4p+4$	$ey+\overline{2p+3}\times yy-\overline{p+3}\,eyy-\overline{p+3}\times$ $\frac{p+2}{2}y^3+3ey^3+y^4$
$4p+5$	$ey+\overline{2p+4}\times yy-\overline{p+4}\,eyy-\overline{p+4}\times$ $\frac{p+3}{2}y^3+6ey^3+4y^4$
$4p+6$	$ey+\overline{2p+5}\times yy-\overline{p+5}\,eyy-\overline{p+5}\times$ $\frac{p+4}{2}y^3+10ey^3+10y^4$
$5p$	$ey+\overline{3p-1}\times yy-\overline{2p-1}\,eyy-\overline{2p-1}$ $\times\frac{2p-2}{2}y^3+\overline{p-1}\times\frac{p-2}{2}ey^3+\overline{p-1}\times$ $\frac{p-2}{2}\times\frac{p-3}{3}y^4$

Example I.

What is the probability of throwing a propoſed chance three times without intermiſſion, in ten trials, once or oftener, when the odds for its happening in any aſſigned trial, is as 2 to 1. Here $a=2$, $b=1$, $n=10$, $d=3$, $p=3$, whence $\overline{\overline{n-p}\times\frac{b}{d}+1}\times\frac{a^p}{d^p}=\frac{80}{81}$, and again $A=4$, $B=1$, $e=\frac{8}{27}$, $y=\frac{8}{81}$, therefore $A\times ey+A\times\frac{A-1}{2}\times yy=\frac{128}{729}$, which being ſubtracted from $\frac{80}{81}$ leaves $\frac{592}{729}$ for the probability in this caſe.

Example II.

In twenty throws with a ſingle die, what is the probability that the ace comes up eight times ſucceſſively. Here $a=1$, $b=5$, $n=20$, $d=6$; $p=8$, therefore $\overline{\overline{n-p}\times\frac{b}{d}+1}\times\frac{a^p}{d^p}$ $=\frac{11}{1679616}$, and A being in this caſe $=4$, $e=\frac{1}{6^8}$, $y=\frac{5}{6^9}$, we have $A\times ey+A\times\frac{A-1}{2}\times yy=,000000000003$, which ſubtracted from the former, leaves the probability required equal to $\frac{1099995}{1679616000000}$ nearly.

Example III.

In one hundred throws with a ſingle die, what is the odds that the ace does not come up five times ſuceeſſively. In this caſe $a=1$,

$b=5$, $n=100$, $d=6$, $p=5$, whence $\overline{n-p\times\frac{b}{d}+1}\times\frac{a^p}{d^p}$ becomes $\frac{481}{6^6}$, and our ſeries $A\times ey+A\times\frac{A-1}{2}\times yy-B\times\frac{B-1}{2}\times eyy$, &c. $=$,000047748 nearly, which ſubtracted from $\frac{481}{6^6}$, leaves .010261252, and this taken from 1, leaves the probability required equal to ,989738748, therefore the required odds will be as ,98973, &c. to ,010261252, or as 98 to 1 nearly.

Note, The ſolution to this example is exhibited extremely near the truth by taking only three terms of the ſeries; and in all caſes where a near approximation is only wanted, a few terms of the ſeries will be ſufficient.

Mr *Demoivre*, at page 243 of the ſecond edition of his *Doctrine of Chances*, gives (but without the inveſtigation) the following method of ſolving the foregoing problem.

Let the probability of winning in any one aſſigned trial be repreſented by $\frac{a}{a+b}$, that of the contrary by $\frac{b}{a+b}$, ſuppoſe *(n)* to repreſent the number of trials given, and *(p)* the number of times to be won ſucceſſively, put $x=\frac{b}{a+b}$, take the quotient of unity divided by $1-x-ax^2-a^2x^3-a^3x^4-a^4x^5-a^5x^6-a^6x^7$

$a^6x^7 \ldots\ldots a^px^p$, and having taken as many terms of the ſeries reſulting from that diviſion, as there are units in $n-p+1$, multiply the ſum of the whole by $\frac{\overline{ax}^p}{b^p}$, or by $\frac{a^p}{\overline{a+b}^p}$, and that product will expreſs the probability required.

Example I.

Let it be required to find the probability of throwing any aſſigned chance three times together, in ten trials, when *(a)* and *(b)* are in a ratio of equality, otherwiſe when each of them is equal to unity, then having divided 1 by $1-x-x^2-x^3$ the quotient continued to ſo many terms as there are units in $n-p+1$, that is, in this caſe to $10-3+1=8$ will be $1+x+2xx+4xxx+7x^4+13x^5+24x^6+44x^7$, where x being interpreted by $\frac{b}{a+b}$, which is in this example $\frac{1}{2}$, the ſeries will become $1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\frac{7}{16}+\frac{13}{32}+\frac{24}{64}+\frac{44}{128}$, of which the ſum is $\frac{520}{128}=\frac{65}{16}$, and this being multiplied by $\frac{a^px^p}{b^p}$ that is, in this caſe, by $\frac{1}{8}$, the product will be $\frac{65}{128}$, and therefore it is ſomething more than equal probability, the aſſigned chance will be thrown three times together, ſome time in ten trials, the odds being 65 to 63.

Example II.

What is the probability that in ſix throws with a ſingle die the ace ſhall come up four times ſucceſſively. In this caſe $a=1$, $b=5$, $n=6$, $p=4$, whence, according to Mr *Demoivre*'s preſcription, take the three firſt terms of the quotient ariſing by dividing unity by $1-x-x^2-x^3-x^4$, *viz.* $1+x+2xx$, in which for x write its value $\frac{5}{6}$, gives $\frac{116}{36}$, this multiplied by $\frac{1}{1296}$ produces $\frac{29}{11664}$, the probability required. If the ſolution to this example had been derived from our method, the anſwer would have been $\frac{1}{486}$, which differs from the former by $\frac{5}{11664}$. But to find the reaſon why theſe two methods do not agree, let us apply Mr *Demoivre*'s theorem to the ſolution of another example or two.

Example III.

Required the probability of winning three times ſucceſſively in ſix trials, the chances for winning at any aſſigned trial being 2, for loſing 1. Here $a=2$, $b=1$, $p=3$, $x=\frac{1}{3}$, and $n=6$, whence, dividing unity by $1-x-2x^2-4x^3$, and taking $n-p+1$, or 4 terms of the quotient, *viz.* $1+x+3xx+9x^3$, equal to $1+\frac{1}{3}+\frac{3}{9}+\frac{9}{27}$, and multiplying the ſum by $\frac{a^p}{\overline{a+b}|^p}$, or $\frac{8}{27}$, gives $\frac{16}{27}$, the probability ſought, and is the very ſame as would come out by the method uſed in the ſolution of *Problem* XLIX. Let us now ſuppoſe $b = 5$, the other quantities remaining as in the example. Now it is very evident, that only the value of x is influenced by this alteration, which in this caſe is $\frac{5}{7}$; therefore the quotient of the diviſion, and the number of terms thereof, that muſt be taken, remain the very ſame as above, whence x in the quotient $1 + x + 3xx + 9x^3$, being interpreted by $\frac{5}{7}$, we have $1 + \frac{5}{7} + \frac{75}{49} + \frac{1125}{343}$, which multiplied by $\frac{8}{343}$, gives $\frac{8904}{117649}$ for the probability required. By our method the probability is $\frac{176}{2401}$, which is leſs than the former by $\frac{40}{16807}$. The multiplier $\frac{a^p}{\overline{a+b}^p}$, being common to both theorems, it ſhould follow, that $\overline{n-p} \times \frac{b}{a+b} + 1$, and $1 + x + 3xx + 9x^3$, are equal to each other; but that this is impoſſible when (b) is greater than unity, is ſo evident it needs no illuſtration.

Example IV.

To find the probability of an event happening three times ſucceſſively, in five trials, the chances for happening in any aſſigned trial being denoted by (a), thoſe for failing by *(b)*. Here if we divide unity by $1-x-axx-axxx$, the quotient will be $1+x+\overline{a+1}\,x^2+\overline{a^2+2a+1}\,x^3+\overline{3aa+3a+1}\,x^4+\overline{2a^3+6aa+4a+1}\,x^5$, &c. which expreſſion is rather more general than any of thoſe deduced from the foregoing diviſions, and ſerves for any value of *(a)*. Now $\overline{1+x+\overline{a+1}\,xx}\times\frac{a^3}{\overline{a+b}^3}$, is the required probability, and by our method the probability of winning three times ſucceſſively, in five trials, is $\overline{\frac{2b}{a+b}+1}\times\frac{a^3}{a+b^3}$, therefore if both methods agree, $1+x+\overline{a\times 1}\,xx$ ſhould be equal to $\frac{2b}{a+b}+1$, for x ſubſtitute its value $\frac{b}{a+b}$, we get $1+\frac{b}{a+b}+\frac{abb+bb}{\overline{a+b}^2}=\frac{2b}{a+b}+1$, this equation properly reduced gives $a=ba$ or $b=1$. Wherefore as the value of *(b)* muſt always be unity, it follows, that in order to have a true ſolution by Mr *Demoivre*'s method, the value of *(b)* in the ratio of the chances for happening or failing muſt be changed into unity. Of this take an example: Let it be required to find the probability of throwing with a ſingle die, the ace three times ſucceſſively, in ſeven throws. Here the chances for failing in any one trial being 5, and for happening only 1, we muſt make $a = \frac{1}{5}$, wherefore taking the firſt five terms of the quotient of unity divided by $1 - x - ax^2 - a^2x^3$, &c. and interpreting (a) by $\frac{1}{5}$, (x) by $\frac{5}{6}$, we ſhall have $1 + \frac{5}{6} + \frac{5}{6} + \frac{5}{6} + \frac{1075}{1296}$, or $\frac{5611}{1296}$, which multiplied by $\frac{1}{216}$ gives $\frac{5611}{289936}$ the required probability, the very ſame as would reſult from our method of ſolution: hence we may infer, that in order to have a true ſolution by Mr *Demoivre*'s method, if (b) is not given equal to unity in the queſtion, it muſt be reduced to it, by taking $\frac{a}{b}$ for a, whence x will become $\frac{b}{a}$, theſe reductions being made, that gentleman's method will always ſucceed.

PROBLEM LI.

Three gameſters, A, B, and C, enter into play together, and they agree that he ſhall be reputed the winner who firſt beats the other two ſucceſſively; alſo that A and B ſhall begin the play, and the loſer ſhall yield his place to the third man, which is conſtantly to be obſerved afterwards; it is required to find the value of their reſpective expectations, ſuppoſing each perſon had ſtaked one guinea.

SOLUTION.

Firſt, it is evident that the probability of the play continuing for more than two games, depends upon that of the gameſters beating each other alternately. Secondly, that becauſe A and B are to begin the play, their expectations will be exactly alike, and it matters not with regard to C, which of them ſhall happen to win the firſt game. Thirdly, that if C beats them both, it muſt be at the end of either the third, ſixth, ninth, twelfth, &c. games: theſe things premiſed, it will be eaſily ſeen that C's probability of receiving the money depoſited, at the end of any aſſigned (poſſible) number of games, is compounded of the probability that C ſhall win the two laſt of thoſe games, and that neither A, B or C ſhall have won two games ſucceſſively before. A's probability of winning the firſt game is $\frac{1}{2}$, that of C's getting the two next $\frac{1}{4}$, whence $\frac{1}{2}\times\frac{1}{4}$ is the probability that both thoſe events may happen. But if B had won the firſt game, ſtill C's probability would have been the ſame, *viz.* $\frac{1}{2}\times\frac{1}{4}$, conſequently $\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4}$ or $\frac{1}{4}$ is C's total probability of winning the ſet, at the end of three games, and by the ſame way of reaſoning, if we ſuppoſe the play will end in ſix games, and not before, his probability will be found equal to $\frac{1}{32}$, now if *n* be put for the number of games at the end of which the ſet will be ended in favour of him, we ſhall have $\frac{1}{2}^{n-3} \times \frac{1}{4}$, his required probability in that caſe, wherefore by writing 3, 6, 9, 12, *&c.* ſucceſſively for *n*, we get this progreſſion $\frac{1}{4} + \frac{1}{32} + \frac{1}{256} - \frac{1}{2048}$, *&c. in infinitum*, whoſe ſum $\frac{4}{14}$ is C's total probability for winning the ſum depoſited, which ſubtracted from unity and the remainder divided by 2, gives $\frac{5}{14}$ for A or B's probability, each of theſe probabilities being multiplied by the ſum depoſited, gives the value of their reſpective expectations, in this caſe, 1*l.* 2*s.* 6*d.* 1*l.* 2*s.* 6*d.* and 18*s.* for A, B, and C.

If the law of the play had been ſuch that each perſon when he yields his place to the next man ſhould be fined a certain ſum *p*, which ſhall ſerve to increaſe the common ſtock, and the winner to have the whole ſum depoſited, then, the other parts remaining as before, the inveſtigation will be as follows.

It appears by the preceding operation that $\frac{1}{2}^{n-3} \times \frac{1}{4}$ is C's probability for winning the ſet at the end of *n* games, but at the end of thoſe games he will be intitled to 3 (the ſum depoſited at firſt) more *np*, wherefore if for

$\frac{1}{2}$ we put y, and for the numerical quantities 3, 6, 9, 12, *&c.* write n, the total expectation of C will be $y^2 \times \overline{3+3p} + y^5 \times \overline{3+6p} + y^8 \times \overline{3+9p} + y^{11} \times \overline{3+12p}$, *&c.* or $3y^2+3y^5+3y^8+3y^{11}+3y^{14}$, *&c.* $+3py^2+6py^5+9py^8+12py^{11}+15py^{14}$, *&c.* The firſt of theſe ſeries is a geometrical progreſſion, whoſe ſum by the common methods is $\frac{y^2}{1-y^3} \times 3$. But to find the ſum of the other, let $y^3+y^6+y^9+y^{12}$, *&c.* $= \frac{y^3}{1-y^3}$ be wrote down, then by taking the fluxion of each ſide the equation, we ſhall have $3y^2\dot{y}+6y^5\dot{y}\,9y^8\dot{y}+12y^{11}\dot{y}$, *&c.* $= \frac{3y^2\dot{y} \times \overline{1-y^3} + 3y^5\dot{y}}{\overline{1-y^3}^2}$, whence by ſtriking out $\dot{y}$ on each ſide the equation, and multiplying the whole by p, the ſecond part of C's expectation will be $3py^2+6py^5+9py^8+12py^{11}$, *&c.* $= \frac{3py^2}{1-y^3} + \frac{3py^5}{\overline{1-p^3}^2}$, to this add the former part $\frac{3y^2}{1-y^3}$, the ſum $\frac{3y^2}{1-y^3} \times \overline{1+p} + \frac{y^5}{\overline{1-y^3}^2} \times 3p$, is the required expectation of C. Now to find A's expectation, let us ſuppoſe that he gets the firſt game, then it is evident that if he wins the ſet afterwards, it muſt be at the end of either the ſecond, fifth, eighth, *&c.* games. But if he is ſuppoſed to loſe the firſt game, and afterwards to win the ſet, it muſt be at the end of either the fourth, ſeventh, tenth, *&c.* games, wherefore y being put for $\frac{1}{2}$ as before, we have $y^2 \times \overline{3+2p} + y^5 \times \overline{3+5p} + y^8 \times \overline{3+8p}$, *&c.* $+ y^4 \times \overline{3+4p} + y^7 \times \overline{3+7p} + y^{10} \times \overline{3+10p}$, *&c.* or by ſumming the ſeries $\frac{\overline{2p+3} \times y^2 + \overline{4p+3} \times y^4}{1-y^3} + \frac{3py^5+3py^7}{\overline{1-y^3}^2}$, for A's required expectation, which is alſo B's, hence by reſtoring the value of y, their reſpective expectations become $\frac{15}{14} + \frac{99}{98}p$, $\frac{15}{14} + \frac{99}{98}p + \frac{12}{14} + \frac{96}{98}p$.

Corollary.

If the common ſtake were conſtant, that is, if there were no fines, then if that conſtant ſtake were called q, their expectations would be $\frac{y^2+y^4}{1-y^3} \times q$, $\frac{y^2+y^4}{1-y^3} \times q$, $\frac{y^2}{1-y^3} \times q$, reſpectively, becauſe in that caſe $p=0$, and the ſum of all the money depoſited equal to q.

The computation of their ſeveral riſks, or the ſum of the probabilities of being fined, together with the value of their ſeveral gains thereon depending, may, by a due conſideration of what has been here delivered, be very eaſily inveſtigated. And if there are four, or a greater number of gameſters, playing together upon the ſame terms as in the problem, the ſolution might be obtained much after the ſame manner; it might perhaps be attended with ſomewhat more labour, becauſe the number of geometrical progreſſions and the other ſeries increaſe with the number of players, yet as the method here given would with very little alteration find the ſum of any ſuch ſeries as could occur, the reader would not be at any loſs how to proceed towards a ſolution of any caſe of the problem whatſoever.

PROBLEM LII.

A and B, whoſe chances for winning any aſſigned game are in the given ratio of *a* to *b*, enter into play on this condition, that A at the beginning of every game ſhall ſet the ſum *c* to the ſum *d*, and that the play ſhall continue as long as A wins ſucceſſively; required the gain or advantage of A.

SOLUTION.

A's expectation upon any one game is $\frac{a}{a+b} \times \overline{c+d}$, from which ſubtract his ſtake leaves $\frac{ad-bc}{a+b}$ for his gain upon that game, but the probability of getting into that circumſtance depends upon his winning all the foregoing games; wherefore if n repreſent any number of games from the beginning of the play, his expectation of gain upon the n^{th} game will be expreſſed by $\frac{a^{n-1}}{\overline{a+b}^{\,n-1}} \times \frac{ad-bc}{a+b}$, wherein if for n be taken ſucceſſively the numbers 1, 2, 3, 4, 5, &c. we ſhall have $\frac{ad-bc}{a+b} \times \overline{1 + \frac{a}{a+b} + \frac{a^2}{\overline{a+b}^2} + \frac{a^3}{\overline{a+b}^3} + \frac{a^4}{\overline{a+b}^4}}$, &c. *ad infinitum*, that is, $\frac{ad-bc}{a+b} \times \frac{a+b}{b}$, or $\frac{ad}{b} - c$, A's gain required.

PROBLEM LIII.

A having three guineas in his hand and B two, agree to throw up each his own pieces, and that he who brings moſt heads uppermoſt, ſhall take all five; and they likewiſe agree, that if both bring up the ſame number of heads, to toſs up again; and ſo to continue till it ſhall appear who is the winner. Quere the proportion of their reſpective chances for winning?

SOLU-

SOLUTION.

Let P, P', P'', be the reſpective probabilities that A has for bringing up 1, 2, 3 heads preciſely, in each throw with three pieces, p, p', p'', thoſe that B has for bringing up either none, one, or two heads preciſely, each time he throws up two pieces. Then $P\times p$ will denote the probability that A ſhall bring up one head preciſely, and at the ſame time B all tails. $P'\times p$ the probability that A brings up two heads, and at the ſame time B all tails. $P''\times p$ that of his bringing up three heads, and B at the ſame time all tails. Thus by collecting together the ſeveral variations that can be made in favour of A, we ſhall have $\overline{P+P'+P''}\times p+\overline{P'+P''}\times p'+P''\times p''$, which being multiplied by $1+y$ (y being put for the probability that the play continues without limitation) gives A's required probability. The probability that B brings up one head preciſely, and at the ſame time A all tails, is $P''\times p'$, that of his bringing up two heads, and at the ſame time A all tails, is $P''\times p''$, to the ſum of theſe two probabilities add that of B's bringing up two heads, and at the ſame time A preciſely one, *viz.* $P\times p''$, the ſum $P''\times p'+P''\times p''+P\times p''$ being multiplied by $1+y$ gives B's required probability. The values of P, P′, P″, p, p', p'', are $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, found by *Problem* VII, therefore A's probability is equal to $\frac{1}{2}\times\overline{1+y}$, B's equal to $\frac{3}{16}\times\overline{1+y}$, whence the ratio of their chances for winning are as 8 to 3, and in this proportion muſt the ſum adventured be divided, conſequently A's expectation is worth 3*l.* 16*s.* 4¼*d.* B's 1*l.* 8*s.* 7¾*d.*

PROBLEM LIV.

Other things remaining as in the laſt problem, ſuppoſe the agreement between the players had been ſuch, that if the play does not terminate in a certain number of games, it ſhall at the end of that number be diſcontinued, and each adventurer take his own money; required their reſpective probabilities for winning.

SOLUTION.

The probability that A brings up preciſely one head at any aſſigned throw is P, that B does the ſame is p', their product, or the probability that both thoſe events happen is $P\times p'$, A's probability for bringing up two heads preciſely, is P′, B's is p'', their product $P'\times p''$. The probability that all the five pieces ſhall come up tails, is $P''\times p''$, therefore the probability that the play does not terminate at any aſſigned throw, is $P\times p'+P'\times p''+P''\times p''$, which put equal to d, and for $\overline{P+P'+P''}\times p+\overline{P'+P''}\times p''+P''\times p''$, and $P''\times p'+P''\times p''+P\times p''$, which are the reſpective probabilities that A and B have for winning at any aſſigned throw, put a, and b, let n repreſent the number of games or throws, beyond which (by their agreement) the play is not to be continued, then $a+da+d^2a+d^3a$, *&c.* . . . . . $d^{n-1}a$, and $b+bd+d^2b+d^3b$, *&c.* . . . . . $d^{n-1}b$, or by ſumming up the progreſſions $\frac{1-d^n}{1-d}\times a$, and $\frac{1-d^n}{1-d}\times b$, are their reſpective probabilities of winning the ſum played for.

Corollary I.

If n is ſuppoſed infinite, or the play to continue without limitation, then d^n becomes infinitely ſmall, and conſequently may be rejected as inconſiderable, whence A's probability in that caſe is $\frac{1}{1-d}\times a$, B's $\frac{1}{1-d}\times b$, and hence the value of y, which was put for the probability of the play continuing without limitation, in the laſt problem, becomes known, it being equal to $d+d^2+d^3+d^4$, *&c. in infinitum*, equal to $\frac{d}{1-d}$.

Corollary II.

The probability that the play does not terminate at the expiration of *(n)* games, being $\frac{d-d^{n+1}}{1-d}$, it is evident that A's expectation thereon is $\frac{d-d^{n+1}}{1-d}\times 3$ and B's $\frac{d-d^{n+1}}{1-d}\times 2$, whence their total expectations are $\frac{1-d^n}{1-d}\times 5a + \frac{d-d^{n+1}}{1-d}\times 3$, and $\frac{1-d^n}{1-d}\times 5b + \frac{d-d^{n+1}}{1-d}\times 2$, reſpectively.

PROBLEM LV.

A and B playing together upon the ſame conditions as in the laſt problem; it is required to find their reſpective probabilities for winning the ſum adventured, ſuppoſing A has *(m)* and B *(n)* guineas.

SOLUTION.

The probability of bringing up preciſely *(p)* heads, in one throw with *(m)* pieces, is $\frac{1 \text{ into } m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}, \&c. \ldots \frac{m-p+1}{p}}{2^m}$, and that of bringing up a leſs number of heads than *(p)* in one throw with *(n)* pieces, is

$$\frac{1+n+n\times\frac{n-1}{2}+n\times\frac{n-1}{2}\times\frac{n-2}{3},\text{\&c. to } (p) \text{ number of terms}}{2^{n}}.$$

Now by taking *p*, ſucceſſively equal to 1, 2, 3, 4, 5, &c. . . . *m*, and multiplying the ſeries thence reſulting together, we ſhall have A's probability for winning (the ſum played for) at any aſſigned throw, equal to

$$\frac{m+m\times\frac{m-1}{2}\times\overline{1+n}+m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\overline{1+n+n\times\frac{n-1}{2}}+}{2^{m+n}}$$

$$\frac{m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\overline{1+n+n\times\frac{n-1}{2}+n\times\frac{n-1}{2}\times\frac{n-2}{3}}}{2^{m+n}}$$

&c. wherein if *n* be wrote for *m*, and *vice verſa*, the ſaid expreſſion will denote B's. Let the probabilities thus determined be called E, and F, reſpectively, and put S for the probability that the play does not terminate at the end of any one throw or game (equal to $1+mn+m\times\frac{m-1}{2}\times n\times\frac{n-1}{2}+m\times\frac{m-1}{2}\times\frac{m-2}{3}\times n\times\frac{n-1}{2}\times\frac{n-2}{3}$, &c. continued to as many terms as there are units plus 1, in the leaſt of the given quantities *m*, *n*, and the whole divided by 2^{m+n}). Then will $\frac{1-S^p}{1-S}\times$ E, and $\frac{1-S^p}{1-S}\times$ F, be their required probabilities for winning the ſum adventured, at or before the expiration of (*p*) games.

Corollary I.

The probability that the play terminates not at the end of *(p)* games nor before, is $\frac{S - S^{p+1}}{1 - S}$, therefore the values of their expectations, before they begin to play, are reſpectively equal to $\frac{1 - S^{p}}{1 - S} \times E \times \overline{m + n} + \frac{S - S^{p+1}}{1 - S} \times m$, and $\frac{1 - S^{p}}{1 - S} \times F \times \overline{m + n} + \frac{S - S^{p+1}}{1 - S} \times n$.

Corollary II.

If *(p)* be very great or infinite, as there can then be but very little or no probability of their withdrawing their ſtakes, or the play terminating before one of them has won the others money, it follows, that the two laſt expreſſions will become $\frac{1}{1 - S} \times E \times \overline{m + n}$, and $\frac{1}{1 - S} \times F \times \overline{m + n}$.

PROBLEM LVI.

To determine at the game of whiſt, the probability that the dealer and his partner have between them the four honours.

SOLUTION.

The dealer being by the law of the game certain of having one trump, the probability of that trump being an honour is $\frac{4}{13}$, becauſe there are 16 cards in his favour, and 36 againſt him. But if the trump turned up ſhould happen to be an honour, their probability of having the other three, will be the ſame as that of taking 3 aſſigned cards in 25, from a pack conſiſting of 51 cards, which, by *Problem* XXXVI, will be found equal to $\frac{92}{833}$, wherefore the probability that both theſe ſhall happen, is $\frac{4}{13} \times \frac{92}{833}$. But if the dealer ſhould happen not to turn up an honour, of which the probability is $\frac{9}{13}$, then their probability of having between them all the four honours, will be the ſame as that of having 4 aſſigned cards in 25, taken from a pack conſiſting of 51 cards, which probability is, by *Problem* XXXVI, equal to $\frac{253}{4998}$, whence $\frac{253}{4998}$ being multiplied by $\frac{9}{13}$, gives their probability of having the four honours, when the card turned up is not an honour, and conſequently $\frac{92}{833} \times \frac{4}{13} + \frac{253}{4998} \times \frac{9}{13}$, is their total probability of having between them the four honours, equal to $\frac{115}{1666}$.

PRO-

PROBLEM LVII.

A and B enter into play together with a pair of common dice, upon theſe conditions, that if A happens to bring up the ſame chance both the firſt and ſecond throw, he wins the game; otherwiſe, A takes the chance firſt thrown, B the next, and he to be reputed the winner whoſe chance ſhall happen firſt to ariſe; from hence it is required to find their reſpective probabilities for winning the game.

SOLUTION.

It is evident that the difference of their required probabilities is equal to the probability of throwing the ſame chance twice ſucceſſively, for if that circumſtance be ſuppoſed omitted, they then play upon equality of chance, and therefore each of their probabilities is equal to $\frac{1}{2}$, conſequently, the probability for throwing the ſame chance twice ſucceſſively, being divided by 2, and the quotient added to $\frac{1}{2}$, the ſum will give A's probability for winning the game. The probability for throwing the chance II. twice ſucceſſively, is $\frac{1}{36} \times \frac{1}{36}$, that for throwing the chance III. twice ſucceſſively, is $\frac{2}{36} \times \frac{2}{36}$, and proceeding in this manner with the remaining

ing chances, we ſhall have the probability of throwing the ſame chance twice ſucceſſively, *viz.* $\frac{2}{1296} + \frac{8}{1296} + \frac{18}{1296} + \frac{32}{1296} + \frac{50}{1296} + \frac{36}{1296}$, equal to $\frac{146}{1296}$, which divided by 2, and the quotient added to $\frac{1}{2}$, gives $\frac{721}{1296}$, A's probability required, whence B's is $\frac{575}{1296}$.

PROBLEM LVIII.

A perſon undertakes to throw with a pair of common dice, the chance VII. before any other ſhall come up twice; required his probability of doing it.

SOLUTION.

Suppoſe the firſt throw (which may be any of the chances upon the dice except VII.) to be II, the probability of producing VII. before II. is $\frac{6}{7}$, which multiplied by the probability of throwing II. the firſt throw, gives $\frac{6}{7 \times 36}$. Suppoſe III. the chance firſt thrown, the probability of producing VII. before III. is $\frac{6}{8}$ or $\frac{3}{4}$, which multiplied by the probability of throwing III. the firſt throw gives $\frac{3}{72}$, whence $\frac{1}{42} \times \frac{1}{24}$ is that part of the required probability depending upon the firſt throw being either II. or III. Proceeding in this manner with the remaining chances, we ſhall have $\overline{\frac{2}{7}+\frac{1}{2}+\frac{2}{3}+\frac{4}{5}+\frac{10}{11}}\times\frac{1}{6}=\frac{7303}{13860}$ for the probability ſought.

PROBLEM LIX.

To find at the game of hazard, the probability of the ſetter for winning the ſum depoſited, upon all the ſuppoſitions of main and chance.

SOLUTION.

In order to facilitate the operation, I have in the table annexed, inſerted the ſcheme of the game.

Mains for the Setter.	Throws next following.	
	Chances.	*Chances.*
	The Caſter wins the ſtake.	The Setter wins the ſtake.
V.	V.	II. III. XI. XII.
VI.	VI. XII.	XI. II. III.
VII.	VII. XI.	XII. II. III.
VIII.	VIII. XII.	XI. II. III.
IX.	X.	II. III. XI. XII.

Let us firſt ſuppoſe VII. to be the main, then the caſter before he begins to throw, has eight chances for the whole ſum depoſited, that is, he has eight chances for VII. and XI, either of which intitles him to that ſum, four chances for nothing, *viz.* thoſe for II. III. XII. ſix chances for $\frac{1}{3}$ of the ſtakes, namely, thoſe for IV. and X. eight for $\frac{2}{5}$ thereof, thoſe for V. and IX. and ten chances for VI. and VIII. theſe laſt intitle him to $\frac{5}{11}$ of the ſtakes, whence if the ſum depoſited be S, the caſter's expectation thereon will be $\frac{8 \times 1 + 6 \times \frac{1}{3} + 8 \times \frac{2}{5} + 10 \times \frac{5}{11}}{36} \times S$, and therefore his required probability of winning the whole ſum depoſited is $\frac{244}{495}$, which being taken from unity leaves $\frac{251}{495}$ for the ſetter's.

Suppoſe VI. or VIII. to be the main, then the chances XI. II. III. intitle the ſetter to the whole ſtake, or ſum depoſited. If V. or IX. ſhould come up, the ſetter's ſhare would be $\frac{5}{9}$ of S, if IV. or X. come up, he will be intitled to $\frac{5}{8}$ of S, and if VIII. happens to be thrown, his expectation will be $\frac{1}{2}$ S, and laſtly, if the chance ſhould happen to be VII. the ſetter's expectation in that caſe would be $\frac{5}{11}$ of S, now theſe ſeveral expectations being reſpectively multiplied by their probabilities of taking place, gives $\frac{5\times 1+8\times\frac{5}{9}+6\times\frac{5}{8}+5\times\frac{1}{2}+6\times\frac{5}{11}}{36}$ $\times$ S, the ſetter's total expectation, and conſequently his required probability equal to $\frac{7295}{14256}$, whence the caſter's is $\frac{6961}{14256}$.

If V. or IX. be the main, then the ſetter's probability for winning the whole ſtake, will by the ſame way of reaſoning, be found equal to $\frac{6\times 1+6\times\frac{4}{7}+4\times\frac{1}{2}+10\times\frac{4}{9}+6\times\frac{2}{5}}{36}=\frac{1439}{2835}$, and the caſter's $\frac{1396}{2835}$.

Hence it appears that in each caſe the ſetter has the advantage, and the moſt ſo, when the main is VI. or VIII.

PROBLEM LX.

To find at hazard the probability that the ſetter has for winning the ſum depoſited, ſuppoſing he be reſolved to ſet upon the firſt main that may happen to be thrown.

SOLUTION.

The probability that V. or IX. may be the main is $\frac{1}{3}$, which multiply by $\frac{1439}{2835}$. The probability that VI. or VIII. may be the main is $\frac{5}{12}$, let this be multiplied by $\frac{7295}{14256}$. The probability that VIII. may be the main is $\frac{1}{4}$, multiply this by $\frac{251}{495}$, the ſum of theſe products will give the ſetter's probability required, that is $\frac{40745942325}{80023204800}$.

Corollary I.

If the caſter and ſetter each ſtake a guinea, the expectation of the latter will be ſuch a part of two guineas as is denoted by the above fraction, and conſequently his gain will be $\frac{1468679850}{80023204800}$, or $\frac{183}{10000}$ nearly, multiplied by 1 guinea, which is about four pence two farthings and $\frac{1}{4}$ in a guinea.

Corollary II.

The probability of a main is to the probability of no main, as 9817 to 10183, or as 96 to 100 nearly.

PROBLEM LXI.

Suppoſing a certain number of halfpence to be thrown up, and all thoſe that come up heads to be taken away, and the remaining ones to be thrown up again, and ſo on in the ſame manner a given number of times, ſtill taking away thoſe each throw which ſhall happen to come up heads; to find the probability of their being all thrown heads in a given number of trials.

SOLU-

SOLUTION.

Let *(p)* be the given number of halfpence, and firſt, ſuppoſe 2 the given number of throws or trials, put $\frac{1}{2}=a$, then $a^p+p\times a^{p+1}+p\times\frac{p-1}{2}\times a^{p+2}+p\times\frac{p-1}{2}\times\frac{p-2}{3}\times a^{p+3}\ldots a^{2p}$, is the probability of throwing *(p)* halfpence all heads (according to the preſcription given in the problem) in two throws, which being divided by a^p is reduced to $1+pa+p\times\frac{p-1}{2}a^2+p\times\frac{p-1}{2}\times\frac{p-2}{3}a^3+p\times\frac{p-1}{2}\times\frac{p-2}{3}\times\frac{p-3}{4}a^4$, &c. . . . a^p, equal to $\overline{1+a}^p$, which multiplied by a^p gives $\overline{a+a^2}^p$, the probability of ſucceeding in two trials. By the very ſame way of reaſoning the probability of ſucceeding in three throws will be found to be $\overline{a+a^2+a^3}^p$, and in four throws $\overline{a+a^2+a^3+a^4}^p$, whence the law of continuation is manifeſt, wherefore putting *(n)* for the propoſed number of throws, we ſhall have $\overline{a+a^2+a^3+a^4,\&c.\ldots a^n}^p$, or by ſumming up the geometrical progreſſion $\frac{\overline{a-a^{n+1}}^p}{1-a}$ equal to $\overline{1-a^n}^p$, becauſe $a=\frac{1}{2}$, for the probability required.

Otherwiſe.

The probability of throwing any number of halfpence all heads in any given number of throws, by toſſing them all up together, is the ſame as that of bringing them all up heads by toſſing them up ſingly the ſame number of times. The probability of bringing up a tail (p) times ſucceſſively with a ſingle halfpenny is a^n, that of the contrary $1-a^n$, which is the probability of throwing one or more heads in thoſe throws, and there being juſt the ſame probability of ſucceeding with each of the halfpence, it follows that $\overline{1-a^n}^p$ will be the probability required, the very ſame as before.

PROBLEM LXII.

Things remaining as in the laſt problem; it is required to find the number of trials neceſſary to make it an equal chance that the event there propoſed may either happen or fail.

SOLUTION.

Let x be the number ſought, then $\overline{1-a^x}^p$ is the probability of throwing p halfpence all heads in x trials, therefore $\overline{1-a^x}^p = a = \frac{1}{2}$, and $1-a^x = a^{\frac{1}{p}}$, whence $x = \frac{\text{Log.}\ \overline{1-a^{\frac{1}{p}}}}{\text{Log.}\ a}$.

PRO-

PROBLEM LXIII.

A and B, whoſe chances for winning any ſingle game, are in the proportion of *(a)* to *(b)*, the former having two crowns, and the latter one, are determined to play together, each ſtaking a crown every game, until one of them has loſt all his money; to find the values of their expectations before they begin to play.

SOLUTION.

Suppoſe A has loſt one game, put x for the value of his chance in that caſe, and $x+y$ for the value of his chance before they begin to play. When A has but one ſtake left he has *(a)* chances for $x+y$, and *(b)* chances for nothing, whence his expectation is equal to $\frac{ax+ay}{a+b}$, therefore $\frac{ax+ay}{a+b}=x$, conſequently $y=\frac{bx}{a}$. Again, it is evident that before they begin to play, A has *(a)* chances to be in poſſeſſion of three crowns, and *(b)* chances for x, in this caſe his expectation is $\frac{3a+bx}{a+b}$, which muſt be equal to $x+y$, or $x+\frac{bx}{a}$, whence $x=\frac{3aa}{aa+ab+bb}$, and $x+y=\frac{aa+ab}{aa+ab+bb}\times 3$, which being taken from 3, leaves $\frac{bb}{aa+ab+bb}\times 3$, which is B's expectation, before they begin to play.

PROBLEM LXIV.

To determine the ſame things as in the laſt problem, ſuppoſing A has three crowns.

SOLUTION.

Put $x+y$, and x, for the values of A's chances when he is ſuppoſed to have loſt one, and two ſtakes reſpectively, $x+y+z$ the value of his chance before they begin to play. If A loſes two ſtakes his expectation is then equal to $\frac{ax+ay}{a+b}$, therefore we have $\frac{ax+ay}{a+b}=x$, whence $y=\frac{bx}{a}$. If A loſes only one ſtake, his expectation is then equal to $\frac{ax+ay+bx}{a+b}$, therefore $\frac{ax+ay+bx}{a+b}=x+y$, whence $z=\frac{by}{a}=\frac{bbx}{aa}$, conſequently $x+y+z=x+\frac{bx}{a}+\frac{bbx}{aa}$. Now it is evident that before they begin to play A has (a) chances to receive 4 crowns, and (b) chances for $x+y$, in this caſe the value of his expectation is $\frac{4a+bx+by}{a+b}$ equal to $x+y+z$, wherein by ſubſtituting for y and z their reſpective values, we get x

$=$

$=\frac{4a^3}{a^3+baa+bba+b^3}$, and thence $x+y+z=\frac{a^3+aab+bba}{a^3+baa+bba+b^3}\times 4$, the value of A's expectation before they begin to play, which being ſubtracted from 4 leaves $\frac{b^3}{a^3+aab+baa+b^3}\times 4$, equal to B's expectation before they begin to play.

Corollary I.

It appears from hence that the numerator and denominator of the fraction which denotes A's probability of winning the whole ſum, do each conſtitute a ſeries of quantities in geometrical progreſſion, whoſe common ratio is $\frac{a}{b}$, wherefore by putting *(n)* for A's number of ſtakes, his expectation will be expreſſed by

$$\frac{a^n+a^{n-1}b+a^{n-2}b^2+a^{n-3}b^3+a^{n-4}b^4, \&c.\,.\,.\,a\times b^{n-1}}{a^n+a^{n-1}b+a^{n-2}b^2+a^{n-3}b^3+a^{n-4}b^4, \&c.\,.\,.\,b^n}$$

$\times\overline{n+1}$, or (by ſumming up the terms of the progreſſions) $\overline{n+1}$ multiplied by $\frac{a^{n+1}-b^n\times a}{a^{n+1}-b^{n+1}}$, whence B's is $\frac{b^n\times a-b^{n+1}}{a^{n+1}-b^{n+1}}$, multiplied by $\overline{n+1}$.

Corollary II.

When $a=b$, A's expectation becomes $\frac{n}{n+1}\times\overline{n+1}$, and B's $\frac{1}{n+1}\times\overline{n+1}$, that is, n and 1, or their reſpective expectations are directly as the ſtakes they have to loſe.

PROBLEM LXV.

A and B, whoſe chances for winning any ſingle game are in the ratio of a to b, the former having n, and the latter m ſtakes, are determined to play together, until one of them has loſt all; to find the values of their expectations before they begin to play.

SOLUTION.

Let x be the value of A's expectation when he has loſt all his ſtakes but one, then by the laſt problem, $x+\frac{bx}{a}+\frac{b^2x}{a^2}+\frac{b^3x}{a^3}\ (n)$, is the value of A's expectation before they begin to play. Now to determine the value of x. By the laſt problem we have $x+\frac{bx}{a}+\frac{b^2x}{a^2}+\frac{b^3x}{a^3}\ (n+m+1)$ for A's expectation when B has but one ſtake left, in this caſe A has (a) chances for the whole ſum $n+m$, and (b) chances for the ſaid expectation, whence, and by ſumming up the progreſſion, we get $x=\frac{\overline{n+m}\times a^{n+m}\times\overline{b-a}}{b^{n+m}-a^{n+m}}$, wherefore $x+\frac{bx}{a}+\frac{b^2x}{a^2}+\frac{b^3x}{a^3}\ (n)$, becomes $\frac{a^m\times\overline{b^n-a^n}}{b^{m+n}-a^{m+n}}$.

$\times\overline{n+m}$, which is A's required expectation, hence B's expectation is $\frac{b^{m+n}-a^{m}\times b^{n}}{b^{m+n}-a^{m+n}}\times\overline{m+n}$.

Corollary I.

When $m=n$, the ratio of their probabilities for winning the whole ſum played for, is that of $a^{m}\times\overline{b^{m}-a^{m}}$ to $b^{2m}-a^{m}\times b^{m}$, or $\overline{b^{m}-a^{m}}\times a^{m}$ to $\overline{b^{m}-a^{m}}\times b^{m}$, that is, as a^{m} to b^{m}.

Corollary II.

When $a=b$, the ratio of their probabilities above mentioned, becomes that of m to n.

Corollary III.

When b is very ſmall with reſpect to a, and m and n large numbers, b^{m} and b^{n} will be inconſiderable in regard of a^{m} and a^{n}, in ſuch caſe the ſaid ratio will be as a^{m} to b^{m}, or as 1 to $\overline{\frac{b}{a}}|^{m}$ nearly.

Corollary IV.

If the agreement between them had been ſuch, that at every game B ſhould ſet the ſtake G to the ſtake L, the advantage or diſadvantage of A may be determined thus: Let the reſpective probabilities that A and B have for winning the whole ſum played for, *viz.* $nG + mL$, be denoted by E and F (theſe probabilities are determined by the ſolution to the problem) then as A is either to receive the ſum nG, or to pay the ſum mL, he has E chances to be in poſſeſſion of the ſum $nG + mL$, and F chances for nothing; whence his expectation is $\frac{E \times nG + E \times mL}{E+F}$, from which ſubtracting L, his ſum adventured, the remainder $\frac{E \times nG - F \times mL}{E+F}$, is his gain.

Note, If the numerical values of the given quantities are ſuch, that the gain of A ſhould happen to be negative, then the advantage is on B's ſide.

Example I.

Suppoſe B to be throwing with two dice, and every time 2 aces come up, A to give him one guinea, and every time an ace and duce B to give A one, and that they agree to continue the play on in this manner till one of them is a winner of a 100 guineas; to find the expectation of each, and the gain of A by this agreement. Becauſe there are two chances for an ace and duce, and only one for 2 aces, a will in this caſe be $= 2$, $b = 1$, and $m = n = 100$, whence $\frac{2^{100}}{2^{100}+1} \times$

200,

200, is A's expectation, and $\frac{1}{2^{100}+1}\times 200$ is B's, $\frac{2^{100}-1}{2^{100}+1}\times 100$ is A's gain.

Example II.

Let the number of ſtakes on either ſide to be won or loſt be equal, and let that number be 5, let there alſo be an equality of ſkill between the gameſters, and ſuppoſe that B ſets at every game a ſtake of one guinea to a ſtake worth but fifteen ſhillings; to find the gain of A. Here $m=n=5$, $G=21$, $L=15$, $E=1$, $F=1$, whence we get $\frac{E\times nG-F\times mL}{E+F}=45$ ſhillings, the gain of A.

From theſe examples, and the foregoing corollaries, it will appear, that if A and B were to play together for a guinea a game, and A has but one ſingle guinea to loſe, but B any number let it be ever ſo large, and A has the odds of 2 to 1, in each game, he will be more likely to win all B's guineas, than to loſe his ſingle one.

Mr *Demoivre* in his *Doctrine of Chances* gives the following method of

SOLUTION.

Let it be ſuppoſed that A has the counters E, F, G, H, *&c.* whoſe number is *(n)*, and that B has the counters I, K, L, M, *&c.* whoſe number is (*m*); moreover let it be ſuppoſed that the counters are the thing played for, and the value of each counter is to the value of the next following one as *a* to *b*, in ſuch manner as that E, F, G, H, I, K, L, be in geometric proportion, this being premiſed, we may ſuppoſe that A and B in every circumſtance of their play may lay down two ſuch counters as may be proportional to the number of chances each has to get a ſingle counter, for in the beginning of the play A may lay down the counter H, which is the loweſt of his counters, and B the counter I, which is his higheſt, but H : I : : *a* : *b*, therefore A and B play upon equal terms. If A beats B, then A may lay down the counter A, which he has juſt got of his adverſary, and B the counter K, but I : K : : *a* : *b*, therefore A and B ſtill play upon equal terms. But if A loſe the firſt time, then A may lay down the counter G, and B the counter H, which he juſt now got of his adverſary, but G : H : : *a* : *b*, and therefore they ſtill play upon equal terms as before, ſo that as long as they play together, they play without advantage or diſadvantage. Now the value of the expectation which A has of getting all the counters of B, is the product of the ſum he expects to win, and the probability of obtaining it, and the ſame holds in reſpect to B, but the expectations of A and B are ſuppoſed equal, therefore the probabilities which they have reſpectively of winning, are reciprocally proportional to the ſums they expect to win, that is, are directly proportional to the ſums they are poſſeſſed of. Whence the probability which A has of winning all the counters of B, is to the probability which B has of winning all the counters of A, as the ſum of the terms E, F, G, H, *&c.* whoſe number is n, to the ſum of all the terms I, K, L, *&c.* whoſe number is m, that is, as $a^m \times \overline{a^n - b^n}$ to $b^n \times \overline{a^m - b^m}$, as will eaſily appear if thoſe terms which are in geometric progreſſion are actually ſummed up by the known methods. Now the probabilities of winning are not influenced by the ſuppoſition here made, of each counter being to the following in the proportion of a to b, and therefore when thoſe counters are of equal value, or rather ſuppoſed of no value, but ſerve only to make the number of ſtakes won or loſt on either ſide, the probabilities of winning will be the ſame as we have aſſigned.

If by agreement B ſets at every game the ſum G to the ſum L, then will the gain or advantage of A, be expreſſed by $m \times a^m \times \overline{a^n - b^n} - n \times b^n \times \overline{a^m - b^m} \times \frac{aG - bL}{a - b}$.

Demonſtration.

Let R and S reſpectively repreſent the probabilities which A and B have of winning all the ſtakes of their adverſary, which probabilities have been juſt now determined. Let us firſt ſuppoſe that the ſums depoſited by A and B are equal, *viz.* G and G, now ſince A is either to win the ſum mG, or loſe the ſum nG, it is plain that the gain of A ought to be eſtimated by $RmG - SnG$: moreover, ſince the ſums depoſited are G and G, and the proportion of their chances to win one game is as a to b, it follows that the gain of A for each individual game is $\frac{aG - bG}{a + b}$; and for the ſame reaſon the gain of each individual game would be $\frac{aG - bL}{a + b}$, if the ſums depoſited by A and B were reſpectively L and G. Let us therefore now ſuppoſe that they are L and G, then in order to find the whole gain of A in this ſecond circumſtance, we may conſider that whether A and B lay down equal or unequal ſtakes, the probabilities which either of them has of winning all the ſtakes of the other ſuffer not thereby any alteration,

teration, and that the play will be of the ſame duration in both circumſtances before it is determined in favour of either: wherefore the gain of each individual game in the firſt caſe, is to the gain of each individual game in the ſecond, as the whole gain of the firſt caſe, to the whole gain of the ſecond, and conſequently the whole gain of the ſecond caſe will be $\overline{Rm - Sn} \times \frac{aG - bL}{a - b}$, or reſtoring the values of R and S, $m \times a^m \times \overline{a^n - b^n} - n \times b^n \overline{a^m - b^m}$ into $\frac{aG - bL}{a - b}$.

If $m = n$, and $a = b$, then the gain of A will be $m^2 \times \frac{G - L}{2}$.

Let the number of ſtakes be unequal, ſo that A be obliged either to win m ſtakes, or loſe n ſtakes, ſuppoſe an equality of chance between the players, then the gain of A will be $m \times n \times \frac{G - L}{2}$.

If the number of ſtakes be equal, but the ſkill of the gameſters unequal, and in the proportion of a to b, then the gain of A will be $\frac{n \times \overline{a^n - b^n}}{a^n + b^n} \times \frac{aG - bL}{a - b}$. So far Mr *Demoivre*.

Theſe concluſions concerning the gain of A do not agree exactly with thoſe derived by our method of ſolution, and the reaſon of it I take

I take to be, becauſe Mr *Demoivre* ſuppoſes, that the gain of A upon each individual game, conſtantly to remain the ſame, *viz.* $\frac{aG-bL}{a+b}$, which would indeed be true were the value of his expectation upon each individual game invariable, but notwithſtanding the ſums they each ſtake every game are conſtantly the ſame, yet as the value of the expectation from which the ſtake muſt be deducted to give the gain, is continually varying, it ſeems inconſiſtent that his gain upon each individual game ſhould be the abovementioned quantity. It may perhaps be objected, that as A has at every game *(a)* chances for receiving the ſum G, and *(b)* chances for paying the ſum L, his expectation thereon is certainly $\frac{aG-bL}{a+b}$, this would be ſo were it in A's power to quit the play when he pleaſed, but by the agreement made between them the play is to continue until one of them has loſt all, therefore no eſtimate of A's gain can be made, but upon ſuppoſition that the play is terminated in his favour, becauſe he certainly muſt either win the ſum mG, or loſe the ſum nG.

PRO-

PROBLEM LXVI.

A perſon undertakes in a certain number (*n*) of throws, with a pair of common dice, not only to throw the chances V. and VI. but V. before VI. with this reſtriction, that if he happens to throw VI. before V. he does not indeed loſe his wager, but is to proceed as if nothing had been done, ſtill deducting ſo many throws as have been vain from the number of throws which he had at firſt given him; to find his probability of winning.

SOLUTION.

Let the chances for VI. coming up at one throw with two common dice be called (*b*), thoſe for V. (*a*), the whole number of chances upon two dice (*s*). For the probability of throwing V. the firſt throw, and VI. the next, that is $\frac{ab}{ss}$, put P, and for the probabilities of VI. in 2, 3, 4, 5, &c. throws, *viz.* $1-\frac{\overline{s-b}^2}{s^2}$, $1-\frac{\overline{s-b}^3}{s^3}$, $1-\frac{\overline{s-b}^4}{s^4}$, $1-\frac{\overline{s-b}^5}{s^5}$, &c. put L, L′, L″, L‴, L⁗, &c. This preparation being made, take $n=2$, in this caſe his probability for winning is P, let $n=3$. Now if he happens to throw V. the firſt trial, he has nothing more to do than to throw VI. in the two following trials; but if he miſſes V. the firſt throw, he is then in the laſt mentioned circumſtance: wherefore when $n=3$, his probability of winning is $\frac{a}{s}\times L+\frac{s-a}{s}\times P$. Let $n=4$, then if he happens to throw V. the firſt trial, and VI. in the three following trials, he wins, but if he happens to miſs V. the firſt throw, his probability of winning will be $\frac{a}{s}\times L+\frac{s-a}{s}\times P$, whence his total probability of winning when $n=4$, is $\frac{a}{s}\times L\times\frac{s-a}{s}+\frac{a}{s}\times L'+\frac{\overline{s-a}^2}{s^2}\times P$, and by the very ſame proceſs when $n=5$, his total probability will be $\frac{a}{s}\times L\times\frac{\overline{s-a}^2}{s^2}+\frac{s-a}{s}\times\frac{a}{s}\times L'+\frac{a}{s}\times L''+\frac{\overline{s-a}^3}{s^3}\times P$. The ſeveral probabilities found in this manner being collected together will ſtand as in the following Table.

in … Trials.	The probabilities of winning
2	P
3	$P \times \frac{s-a}{s} + L \times \frac{a}{s}$
4	$P \times \frac{\overline{s-a}^2}{s^2} + L \times \frac{s-a}{s} \times \frac{a}{s} + L' \times \frac{a}{s}$
5	$P \times \frac{\overline{s-a}^3}{s^3} + L \times \frac{\overline{s-a}^2}{s^2} \times \frac{a}{s} + L' \times \frac{s-a}{s} \times \frac{a}{s} + L'' \times \frac{a}{s}$
6	$P \times \frac{\overline{s-a}^4}{s^4} + L \times \frac{\overline{s-a}^3}{s^3} \times \frac{a}{s} + L' \times \frac{\overline{s-a}^2}{s^2} \times \frac{a}{s} + L'' \times \frac{s-a}{s} \times \frac{a}{s} + L''' \times \frac{a}{s}$
&c.	&c. &c. &c.

From hence the law of continuation is eaſily found, the exponent of $\frac{s-a}{s}$ being $n-2$, and the number of terms which compoſe the required probability is $n-1$, wherefore that probability, by reſtoring the values of L, L′, L″, L‴, L⁗, &c. will be expreſſed by $\frac{\overline{s-a}^{n-3}}{s^{n-3}} \times \frac{a}{s} + \frac{\overline{s-a}^{n-4}}{s^{n-4}} \times \frac{a}{s} + \frac{\overline{s-a}^{n-5}}{s^{n-5}} \times \frac{a}{s} + \frac{\overline{s-a}^{n-6}}{s^{n-6}} \times \frac{a}{s}$, &c. $(n-2)$ minus $\frac{a}{s^n}$ into the ſum of $\overline{s-a}^{n-3} \times \overline{s-b}^2 + \overline{s-a}^{n-4} \times \overline{s-b}^3 + \overline{s-a}^{n-5} \times \overline{s-b}^4 + \overline{s-a}^{n-6} \times \overline{s-b}^5$, &c. $(n-2)$ plus $\frac{\overline{s-a}^{n-2}}{s^{n-2}} \times P$, wherefore by ſumming up the progreſſions, and ſubtracting the latter from the former, we get his required probability of winning, equal to $\frac{\overline{s-a}^{n-2}}{s^{n-2}} \times P + \frac{s^{n-2} - \overline{s-a}^{n-2}}{s^{n-2}} - \frac{a}{s^n} \times \frac{\overline{s-b}^n - \overline{s-a}^{n-3} \times s \times \overline{s-b}^2 + \overline{s-a}^{n-3} \times \overline{s-b}^2 \times a}{a-b}$

Corollary.

If $a=b$, the aforeſaid probability becomes $\frac{\overline{s-a}^{n-2}}{s^{n-2}} \times P + \frac{s^{n-2} - \overline{s-a}^{n-2}}{s^{n-2}} - \frac{a}{s^n} \times \overline{s-a}^{n-1}$.

Now for the application to numbers: Let $n=4$, and becauſe $a=4$, $b=5$; $s=36$, the required probability will be equal to $\frac{32705}{419904}$, which being ſubtracted from unity leaves $\frac{387199}{419904}$, and the odds againſt throwing the chances propoſed according to the conditions in the problem, in four throws, as 12 to 1 nearly.

If it were undertaken to throw with a pair of common dice in four trials, not only the chances VI. and VIII. but alſo VI. before VIII. the probability of ſucceeding will be found equal to $\frac{154615}{839808}$.

PROBLEM LXVII.

To find the number of chances for throwing (p) points preciſely, with (n) dice, each die having (f) faces, but with this reſtriction, that the propoſed points ſhall come up without aces.

SOLUTION.

Conceive that face upon each die which is marked with (f) points expunged, then by *Problem* XLV. the chances for (p) points preciſely, with the dice ſo reduced, is equal to $\frac{p-1}{1}\times\frac{p-2}{2}\times\frac{p-3}{3}(n-1)-\frac{a-1}{1}\times\frac{a-2}{2}\times\frac{a-3}{3}(n-1)$

into $n + \frac{b-1}{1} \times \frac{b-2}{2} \times \frac{b-3}{3}\,(n-1)$ into $n \times \frac{n-1}{2}$, &c. wherein, $a = p - f + 1$, $b = p - 2f + 2$, $c = p - 3f + 3$, &c. Now if we ſuppoſe the number of points upon each face of every die to be encreaſed by unity, it is evident, that thoſe faces which were before marked I, will now be marked II, and thoſe which were marked II, will now be marked III, and ſo of the reſt, and conſequently the aces are all expunged. Now as every throw which before gave (p) points, will in this caſe give $p + n$, it follows, that if for p in the above expreſſion we write $p - n$, we ſhall have the chances required, equal to

$$\frac{p-n-1}{1} \times \frac{p-n-2}{2} \times \frac{p-n-3}{3}\,(n-1)$$
$$- \frac{a-1}{1} \times \frac{a-2}{2} \times \frac{a-3}{3}\,(n-1) \text{ into } n$$
$$+ \frac{b-1}{1} \times \frac{b-2}{2} \times \frac{b-3}{3}\,(n-1) \text{ into } n \times \frac{n-1}{2}$$
$$\text{\&c.} \qquad\qquad \text{\&c.}$$

wherein $a = p - n - f + 1$, $b = p - n - 2f + 2$, $c = p - n - 3f + 3$, &c.

Example I.

How many chances are there for throwing preciſely 12 points with four common dice, under the above mentioned reſtriction. Here $p = 12$, $f = 6$, $n = 4$, whence the general expreſſion becomes $\frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$.

Example II.

To determine the number of chances for throwing preciſely 20 points with four common dice, excluding aces. Here $p = 20$, $f = 6$, $n = 4$, hence we get $\frac{15}{1} \times \frac{14}{2} \times \frac{13}{3} - \frac{4}{1} \times \frac{10}{1} \times \frac{9}{2} \times \frac{8}{3} + \frac{5}{1} \times \frac{4}{2} \times \frac{3}{3} \times \frac{6}{1}$ for the number required equal to 35. But the number of chances for 20 points with four dice, without reſtriction, is alſo 35, whence it appears, that if 20 points are thrown with four dice, the aces muſt of neceſſity be reſtrained from appearing, which is alſo evident for this reaſon, that no more than 18 points can be thrown with three dice.

PROBLEM LXVIII.

Three perſons, A, B, and C, enter into play together with a pair of common dice, they agree that each perſon ſhall throw in turn, and he to be reputed the winner who firſt happens to bring up the greateſt number of points at one throw. Now ſuppoſing A to have thrown the number VIII; it is required to find (before the others begin to throw) his expectation upon the ſum depoſited, which is here ſuppoſed S.

SOLUTION.

The probability that B or C do not bring up ſo good a chance as A's is $\frac{21}{36}$, wherefore A's expectation upon beating both B and C, is $\frac{21}{36} \times \frac{21}{36} \times S$. Again, the probability that B throws the ſame chance that A has thrown, and C an inferior one, is $\frac{5}{36} \times \frac{21}{36}$, if this ſhould happen A and B will be upon equality of chance, and the expectation of the former equal to $\frac{5}{36} \times \frac{21}{36} \times \frac{S}{2}$. But as with equal probability C might have thrown the ſame chance with A, and B an inferior one, A's expectation in that caſe would have been alſo $\frac{5}{36} \times \frac{21}{36} \times \frac{S}{2}$. Now if B and C ſhould happen each to throw the ſame chance that A has, of which the probability is $\frac{5}{36} \times \frac{5}{36}$, A will be intitled to $\frac{1}{3}$ of the ſtake, wherefore $\frac{S}{3}$ being multiplied by $\frac{5}{36} \times \frac{5}{36}$, and that product added to the ſum of the former expectations, gives A's total expectation $= \frac{1663}{3888} \times S$.

PROBLEM LXIX.

Four perſons, A, B, C, and D, playing together at raffles, or any other game of the like nature, ſuppoſe the firſt of them A to have raffled and got ſuch a number, that there is only the probability *(a)* for a greater,

ter, the probability (b) for the ſame, and the probability (c) for a leſſer number; it is required to find his probability of beating all the other players B, C, and D.

SOLUTION.

Firſt, the probability that B, C, and D, do each bring up inferior chances to A, is c^3, this is one part of A's probability for winning the game. Secondly, if B throws the ſame chance that A has thrown, and at the ſame time C and D inferior chances, of which the probability is $a\times c^2$, A and B will be upon equality of chance, and either of their probabilities equal to $a\times c^2\times\frac{1}{2}$. But as this circumſtance might with equal probability have happened to C or D inſtead of B, it follows, that the ſecond part of A's probability is equal to $3\times a\times c^2\times\frac{1}{2}$. Thirdly, if B and C, or any other two of the players, ſhould happen to come upon equality of chance with A, and at the ſame time the other player be excluded the play, of which the probability is $3\times\frac{3-1}{2}\times b^2\times c$, then will A's probability be equal to $\frac{1}{3}\times 3\times\frac{3-1}{2}\times$ $\times b^2\times c$. Laſtly, if B, C, and D, ſhould each happen to throw the ſame chance with A, then in that caſe they will all four be upon equality of chance, and A's probability equal to $\frac{1}{4} \times b^3$, conſequently A's total probability for beating the other players B, C, and D, is expreſſed by $c^3 + ac^2 \frac{1}{2} + \frac{1}{3} \times 3 \times \frac{3-1}{2} \times b^2 + \frac{1}{4} b^3$.

PROBLEM LXX.

The other things remaining as in the laſt problem. Let the number of players be called $n+1$, to determine A's probability of beating all his adverſaries.

SOLUTION.

The probability that A's chance already thrown ſhall be better than any chance the other players ſhall happen to throw, is c^n. That ſome one of them ſhall come upon equality with A, and all the others excluded the play, is $n \times b \times c^{n-1}$, which intitles A to $\frac{1}{2}$ the ſum depoſited. The probability that two of the players may happen to come upon equality of chance with A, and all the remaining players excluded the play, is $n \times \frac{n-1}{2} \times b^2 \times c^{n-2}$, which intitles him to $\frac{1}{3}$ of the ſum depoſited. Again, the probability that three of the players ſhall each throw the ſame chance with A, and the others inferior chances, is $n\times\frac{n-1}{2}\times\frac{n-2}{3}\times b^3\times c^{n-3}$, this multiplied by $\frac{1}{4}$ gives A's probability of winning the ſum depoſited, when all the other players but three are excluded the play. From hence the law of continuation is manifeſt, and the total probability that A has for winning the ſum depoſited, will be expreſſed by $c^n+\frac{n\times c^{n-1}\times b}{2}+\frac{n\times\frac{n-1}{2}\times c^{n-2}\times b^2}{3}+\frac{n\times\frac{n-1}{2}\times\frac{n-2}{3}\times c^{n-3}\times b^3}{4}$, &c. . $\frac{b^n}{n}$. But in order to find the ſum of this ſeries it may be neceſſary to obſerve, that the ſum of the numerators of the fractions that conſtitute it, being equal to $\overline{c+b}^n$, if we put A, B, C, D, &c. for c^n, $n\times c^{n-1}$, $n\times\frac{n-1}{2}\times c^{n-2}$, $n\times\frac{n-1}{2}\times\frac{n-2}{3}\times c^{n-3}$, &c. reſpectively, we ſhall have $A+Bb+Cb^2+Db^3$, &c.... $\frac{b^n}{n}=\overline{c+b}^n$. Now let b be ſuppoſed to become a flowing quantity, and let each ſide of the equation be multiplied by $\dot{b}$, from whence we get $A\dot{b}+Bb\dot{b}+Cb^2\dot{b}+Db^3\dot{b}$, &c. .... $b^n\dot{b}=\overline{c+b}^n\dot{b}$, and by taking the fluent of the equation there ariſes $Ab+\frac{Bb^2}{2}+\frac{Cb^3}{3}+\frac{Db^4}{4}$, &c. . $\frac{b^{n+1}}{n}=$

$\frac{\overline{c+b}^{n+1}}{n+1}$, wherein by making $b=0$, we get $\frac{c^{n+1}}{n+1}=0$, which being evidently impoſſible, it appears that there is wanting the conſtant quantity $\frac{c^{n+1}}{n+1}$, to compleat the fluent, which being added to the ſeries, and the values of A, B, C, D, &c. reſtored, we ſhall have $c^n + \frac{n\times c^{n-1}\times b}{2} + \frac{n\times\frac{n-1}{2}\times c^{n-2}\times b^2}{3}$, &c. . . . $\frac{b^n}{n}$ $+ \frac{c^{n+1}}{\overline{n+1}\times b} = \frac{\overline{c+b}^{n+1}}{\overline{n+1}\times b}$, or by tranſpoſing $\frac{c^{n+1}}{\overline{n+1}\times b}$, A's probability is equal to $\frac{\overline{c+b}^{n+1}}{\overline{n+1}\times b} - \frac{c^{n+1}}{\overline{n+1}\times b}$.

PROBLEM LXXI.

Things ſtill remaining as before, let it be required to find the number of players neceſſary to make A's expectation the greateſt poſſible, ſuppoſing that each adventurer ſtakes a certain ſum S.

SOLU-

SOLUTION.

It appears by the concluſion of the laſt problem, that $\frac{\overline{c+b}^{n+1}-c^{n+1}}{\overline{n+1}\times b}$ will expreſs A's probability for beating all his adverſaries whoſe number is n, from whence the value of his expectation is $\frac{\overline{c+b}^{n+1}-c^{n+1}}{\overline{n+1}\times b}\times\overline{n+1}\times S$, this by the nature of the problem muſt be a maximum, and therefore the fluxion of $\overline{c+b}^{n+1}-c^{n+1}$, muſt be equal to nothing. But to get the fluxion thereof; let $n+1$ be called x, and put $c+b=d$, $d^{x}=y$, $c^{x}=z$. Now $x\times Log.\,d=Log.\,y$, and $x\times Log.\,c=Log.\,z$, wherefore by taking the fluxions of theſe equations, we get $\dot{x}\times Log.\,d=\frac{\dot{y}}{y}$, and $\dot{x}\times Log.\,c=\frac{\dot{z}}{z}$, conſequently $\dot{y}=\dot{x}y\times Log.\,d$, $\dot{z}=\dot{x}z\times Log.\,c$, and therefore the fluxion of $\overline{c+b}^{n+1}-c^{n+1}$ becomes $\dot{n}\times\frac{Log.\,d}{c^{n+1}}-\frac{\dot{n}\times Log.\,c}{d^{n+1}}$, which muſt be $=0$, hence $Log.\,d\times d^{n+1}=Log.\,c\times c^{n+1}$, whence $\frac{d^{n+1}}{c^{n+1}}=\frac{Log.\,c}{Log.\,d}$, now put q for the quotient of the hyperbolic logarithm of c, divided by the hyperbolic logarithm of d, then $\frac{d}{c}\Big|^{n+1} = q$, and $n + 1 = \frac{Log.\, q.}{Log.\, d - Log.\, c}$.

Example I.

Suppoſe A to have thrown the number IX. with a pair of common dice, then $b = \frac{4}{36}$, $c = \frac{26}{36}$, $d = \frac{30}{36}$, $q = ,99373$, and thence $n + 1 = 160$, conſequently the greateſt advantage that A can poſſibly have will be when there are 159 players beſides himſelf.

Example II.

Suppoſe a perſon at play to have ſuch a chance, that there are only the probability $\frac{1}{102}$ for a greater, $\frac{1}{102}$ for the ſame, and $\frac{100}{102}$ for a leſſer. Here $b = \frac{1}{102}$, $c = \frac{100}{102}$, $d = \frac{101}{102}$, whence $q = ,99956$, and $n + 1 = 231$. In this caſe his greateſt advantage will be when there are 230 players beſides himſelf.

PROBLEM LXXII.

Two perſons, A and B, (whoſe chances for winning any aſſigned game are in the ratio of a to b) each having a certain number (p) of guineas, agree to play together upon theſe conditions, *viz.* that A ſhall ſtake *(p)* guineas to one each game, and that the play ſhall continue until either A is incapable to ſet *(p)* guineas to one, or elſe one of them has loſt all his money; from hence it is required to find their reſpective expectations before they begin to play.

SOLUTION.

If A wins the firſt game and loſes the ſecond, of which the probability is $\frac{a}{a+b}\times\frac{b}{a+b}$ the play ends, and A has one guinea left, his expectation in this caſe is $1\times\frac{a}{a+b}\times\frac{b}{a+b}$. Again, if he ſhould happen to win the firſt and ſecond game, and loſe the third, the play will then terminate, and A will have two guineas left, the probability that this circumſtance takes place is $\frac{a^2}{\overline{a+b}^2}\times\frac{b}{a+b}$, whence $2\times\frac{a^2}{\overline{a+b}^2}\times\frac{b}{a+b}$ is his expectation. If the play is ſuppoſed to terminate at the end of four games, the value of A's expectation will in that caſe be equal to $\frac{a^3}{\overline{a+b}^3}\times\frac{b}{a+b}\times 3$, and if A happens to win four games ſucceſſively and loſes the fifth, his expectation will be $\frac{a^4}{\overline{a+b}^4}$

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$\times \frac{b}{a+b} \times 4$, from hence the law of continuation is manifeſt, and by putting $\frac{a}{a+b} = x$, we ſhall have A's expectation upon winning ſucceſſively 1, 2, 3, 4, &c. to $p-1$ games, and loſing the next following, which terminates the play, equal to $x + 2x^2 + 3x^3 + 4x^4$, &c.... $\overline{p-1} \times x^{p-1}$ into $\frac{b}{a+b}$, to this add A's expectation upon winning p times ſucceſſively, *viz.* $2p \times x^p$, and the ſum is his total expectation, equal to $2p \times x^p + \frac{a}{b+a} \times x + 2x^2 + 3x^3 + 4x^4$, &c..... $p-1 \times x^{p-1}$. But to find the ſum of $x + 2x^2 + 3x^3 + 4x^4$, &c.... $p-1 \times x^{p-1}$, let the ſum of $x + x^2 + x^3 + x^4$, &c.... x^{p-1}, be found, which being a geometrical progreſſion, we have $x + x^2 + x^3 + x^4$, &c. ...... $x^{p-1} = \frac{x - x^p}{1-x}$, and by taking the fluxion of each ſide the equation we get $\dot{x} + 2x\dot{x} + 3x^2\dot{x} + 4x^3\dot{x}$, &c.... $p-1 \times x^{p-2}\dot{x} = \frac{\dot{x} - px^{p-1}\dot{x} + \overline{p-1}\, x^p \dot{x}}{\overline{1-x}^2}$, divide each ſide of the equation by $\dot{x}$, and multiply the whole by x, there ariſes $x + 2xx + 3x^3 + 4x^4$, &c. .. $p-1$

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$\times x^{p-1} = \frac{x - px^p + \overline{p-1}\,x^{p+1}}{\overline{1-x}^2}$, which being multiplied by $\frac{b}{a+b}$, and the product added to $2p \times x^p$, the ſum, by reſtoring the value of x, will be equal to $\frac{2p \times a^p}{\overline{a+b}^p} + \frac{a}{b} - \frac{p \times a^p}{\overline{a+b}^{p-1} \times b} + \frac{\overline{p-1} \times a^{p+1}}{\overline{a+b}^p \times b}$, A's total expectation before they begin to play, which ſubtracted from $2p$ leaves B's.

If $a = b$, A's expectation becomes $\frac{p - 1 + 2^p}{2^p}$ and B's $\frac{p \times 2^{p+1} - 2^p + 1 - p}{2^p}$.

Example I.

A and B each having three guineas, play together with a pair of common dice, A agrees to give B three guineas if the aces come up the firſt throw, or to receive one guinea of B in caſe VII. comes up at that throw, and ſo to continue the play according to the conditions mentioned in the problem; to determine their reſpective expectations upon this agreement. Here $p = 3$, $a = 6$, $b = 1$, whence A's expectation becomes $4, \frac{238}{343}$ guineas, and B's $1, \frac{105}{343}$ guineas.

Example

Example II.

If A and B play together upon equality of ſkill, each having ten guineas; what are the reſpective values of their expectations? Here $a=b=1$, $p=10$, whence A's expectation is worth no more than $1,\frac{9}{256}$, but B's is worth $18,\frac{247}{256}$ guineas.

F I N I S.